WS1.5 Dimensional analysis Name: ______________________

(Sections 1.3, 1.4)

The physical problem of a particle bound to a horizontal surface by the (very weak) force of gravity has been explored experimentally in "*Quantum states of neutrons in the Earth's gravitational Field*", V. V. Nesvizhevsky *et al.*, Nature **415**, 297-299 (2002). For this problem, there are only three relevant physical parameters, the mass of the neutron, m, the (very familiar) acceleration of gravity, $g = 9.8\, m/s^2$, and Planck's constant, $\hbar$. The mass of the neutron is only slightly more than that of the proton, namely $m_n \approx 940\, MeV/c^2$ or $1.7 \times 10^{-27}\, kg$.

(a) Using only dimensional analysis, find the product of powers of $m, g, \hbar$ which give an energy, E. Namely, write $E \sim m^\alpha g^\beta \hbar^\gamma$ and solve for α, β, γ as in Sec. 1.4.This would then be the order-of-magnitude of the ground state energy of the system.

(b) Repeat part (a) to find those combinations which combine to describe the appropriate length, time, and speed (l, t, v) scales for this problem.

(c) Evaluate the values of E, l, t, v you obtained numerically, using the physical values for $m, g, \hbar$.

(d) Compare your values (in tabular form) to the corresponding physical quantities for the ground state of the hydrogen atom, as outlined in Sec. 1.3. (For example, the magnitude of the ground state energy of hydrogen is $13.6\, eV$.

Solution: The dimensions of the three physical quantities $m, g, \hbar$, in terms of basic M, L, T labels, are

$$[m] = M \qquad [g] = \frac{L}{T^2} \qquad [\hbar] = \frac{ML^2}{T}$$

Matching the dimensions, we then have

$$[E] = [m]^\alpha [g]^\beta [\hbar]^\gamma \qquad \text{or} \qquad \frac{ML^2}{T^2} = [M]^\alpha \left[\frac{L}{T^2}\right]^\beta \left[\frac{ML^2}{T}\right]^\gamma$$

Comparing powers of M, L, T, we find three equations in three unknowns,

$$\begin{aligned} M: \quad 1 &= \alpha + 0 + \gamma \\ L: \quad 2 &= 0 + \beta + 2\gamma \\ T: \quad -2 &= 0 - 2\beta - \gamma \end{aligned}$$

which can be solved to obtain

$$\alpha = \frac{1}{3} \quad , \quad \beta = \frac{2}{3} \quad , \quad \gamma = \frac{2}{3} \qquad \text{or} \qquad E \sim (mg^2\hbar^2)^{1/3}$$

In similar fashion, one finds the results

$$l \sim \left(\frac{\hbar^2}{m^2 g}\right)^{1/3} \qquad t \sim \left(\frac{\hbar}{mg^2}\right)^{1/3} \qquad v \sim \left(\frac{g\hbar}{m}\right)^{1/3}$$

and one can perform cross-checks such as noting that E and mgl have the same dimensions, as do l/t and v.

For comparison to the hydrogen-atom case, we consult the discussion of Sec. 1.3 and use the bound state energy of roughly $|E| \approx 13.6\, eV$, the Bohr radius value of $l \sim a_0 \approx 0.5\mathring{A}$, the speed $v \sim \alpha c \approx (3 \times 10^8\, m/s)(1/137) \approx 2 \times 10^6\, m/s$, and a time given by $t = 2\pi a_0/v \approx 1.5 \times 10^{-16}\, s$.

For the gravitational bound state of a neutron, we estimate

$$E \sim (mg^2\hbar^2)^{1/3} \approx 1.2 \times 10^{-31}\, J \approx 0.7 \times 10^{-12}\, eV \sim 1\, peV$$

and other order-of-magnitude values as shown in the Table below.

quantity	gravitationally bound neutron	hydrogen atom
E	$1 \times 10^{-12}\, eV$	$13.6\, eV$
l	$10\, \mu m$	$0.5\, \mathring{A}$
t	$1\, ms$	$1.5 \times 10^{-16}\, s$
v	$0.01\, m/s$	$2 \times 10^6\, m/s$