Define
$$A = \int d^3\vec{x}\; x_1 \psi^\dagger(\vec{x},t)|0\rangle\langle 0|\psi(\vec{x},t)$$

(a) ψ acting on a state of n particles gives a state of $n-1$ particles. So, $\langle 0|\psi|\text{n-particle state}\rangle = 0$ unless $n=1$

(b) Let $|f\rangle = \int d^3\vec{x}\, \psi^\dagger(\vec{x},t)|0\rangle f(\vec{x})$

$$A|f\rangle = \int d^3\vec{x}\int d^3\vec{y}\; y_1\, \psi^\dagger(\vec{y},t)|0\rangle\langle 0|\psi(\vec{y},t)\psi^\dagger(\vec{x},t)|0\rangle f(\vec{x})$$

Now $\psi(\vec{y},t)\psi^\dagger(\vec{x},t)|0\rangle = \psi^\dagger(\vec{x},t)\psi(\vec{y},t)|0\rangle + \delta^{(3)}(\vec{x}-\vec{y})|0\rangle$

by ETCCR

$$= 0 + \delta^{(3)}(\vec{x}-\vec{y})|0\rangle.$$

$$\therefore A|f\rangle = \int d^3\vec{x}\, d^3\vec{y}\; y_1\, \psi^\dagger(\vec{y},t)|0\rangle\, \delta^{(3)}(\vec{x}-\vec{y})\, f(\vec{x})$$
$$= \int d^3\vec{x}\; x_1\, \psi^\dagger(\vec{x},t)|0\rangle f(\vec{x}),$$

i.e. a state with wave function $x_1 f(\vec{x})$ at time t.

Hence A is the operator for the first component of position.