Let $f(B) = \sum_n a_n B^n$. (1)

We first show $[A, B^n] = c n B^{n-1}$. (2)

This we do by induction

$n=0$. $[A, B^0] = [A, I] = 0$, so (2) is true for $n=0$.

Suppose (2) is true for $n=n_0$, then

$$[A, B^{n_0+1}] = AB^{n_0+1} - B^{n_0+1}A$$
$$= ABB^{n_0} - BAB^{n_0} + BAB^{n_0} - BB^{n_0}A$$
$$= [A, B]B^{n_0} + B[A, B^{n_0}]$$
$$= cB^{n_0} + Bcn_0B^{n_0-1}$$
$$= c(n_0+1)B^{n_0}.$$

So (2) is true for $n = n_0+1$.

Hence by induction, it is true for all integer $n \geq 0$.

Hence

$$[A, f(B)] = [A, \sum_n a_n B^n]$$
$$= \sum_n a_n [A, B^n]$$
$$= \sum_n a_n c B^{n-1}$$
$$= c \frac{\partial f(B)}{\partial B}$$