Solution to: problem set 3, question 3

The Lagrangian has the form

$$L = \tfrac{1}{2}\sum_{i,j} \dot{q}^i K_{ij} \dot{q}^j - \tfrac{1}{2}\sum_{i,j} q^i V_{ij} q^j \tag{1}$$

$$= \tfrac{1}{2}\dot{q}^T K \dot{q} - \tfrac{1}{2} q^T V q \tag{2}$$

The momenta are

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = \sum_j \tfrac{1}{2} K_{ij}\dot{q}^j + \sum_j \tfrac{1}{2}\dot{q}^j K_{ji} = \sum_j K_{ij}\dot{q}^j, \tag{3}$$

since K is symmetric.

Now K^{-1} obeys $\sum_j (K^{-1})^{ij} K_{jk} = \delta^i_k$, and is symmetric. So

$$\dot{q}^i = \sum_j (K^{-1})^{ij} p_j. \tag{4}$$

Hence the Hamiltonian is

$$\begin{aligned} H &= \sum_i p_i \dot{q}^i - L \\ &= \sum p_i (K^{-1})^{ij} p_j - \tfrac{1}{2}\sum p_{i'} (K^{-1})^{i'i} K_{ij} (K^{-1})^{jj'} p_{j'} + \tfrac{1}{2}\sum q^i V_{ij} q^j \\ &= \tfrac{1}{2}\sum p_i (K^{-1})^{ij} p_j + \tfrac{1}{2}\sum q^i V_{ij} q^j \\ &= \tfrac{1}{2} p^T K^{-1} p + \tfrac{1}{2}\sum q^T V q. \end{aligned} \tag{5}$$

The ETCCR are

$$[q^i(t), q^j(t)] = 0 = [p_i(t), p_j(t)], \qquad [q^i(t), p_j(t)] = i\hbar \delta^i_j. \tag{6}$$

(a) We now change variables to

$$x^i = \sum_j S^i{}_j q^j, \qquad \text{i.e., } x = Sq. \tag{7}$$

Then

$$\dot{x} = S\dot{q}, \qquad q = S^{-1}x, \qquad \dot{q} = S^{-1}\dot{x}. \tag{8}$$

Note that S^{-1} obeys $\sum_j (S^{-1})^i{}_j S^j{}_k = \delta^i_k$. Writing the Lagrangian in the new coordinates gives

$$L = \tfrac{1}{2}\sum (S^{-1})^i{}_{i'} \dot{x}^{i'} K_{ij} (S^{-1})^j{}_{j'} \dot{x}^{j'} - \tfrac{1}{2}\sum (S^{-1})^i{}_{i'} x^{i'} V_{ij} (S^{-1})^j{}_{j'} x^{j'} \tag{9}$$

$$= \tfrac{1}{2}\sum \dot{x}^i \hat{K}_{ij} \dot{x}^j - \tfrac{1}{2}\sum x^i \hat{V}_{ij} x^j, \tag{10}$$

where $\hat{K} = (S^{-1})^T K S^{-1}$ and $\hat{V} = (S^{-1})^T V S^{-1}$.

(b) The momenta for the new coordinates are

$$\begin{aligned}\pi_i = \frac{\partial L}{\partial \dot{x}^i} &= \sum \hat{K}_{ij}\dot{x}^j \\ &= \sum (S^{-1})^{i'}{}_i K_{i'j'} (S^{-1})^{j'}{}_j \dot{x}^j \\ &= \sum (S^{-1})^{i'}{}_i K_{i'j'} \dot{q}^{j'} \\ &= \sum p_{i'} (S^{-1})^{i'}{}_i, \end{aligned} \tag{11}$$

i.e., $\pi = pS^{-1}$, where it is convenient to treat π and p as *row* vectors.

(c) We now the usual rule for finding the Hamiltonian, working with the x and π variables:

$$H = \sum_i \pi_i \dot{x}^i - L = \tfrac{1}{2} \sum \pi_i (\hat{K}^{-1})^{ij} \pi_j + \tfrac{1}{2} \sum x^i \hat{V}_{ij} x^j \tag{12}$$

Now

$$(\hat{K})^{-1} = \left((S^{-1})^T K S^{-1} \right)^{-1} = S K^{-1} S^T, \tag{13}$$

so

$$H = \tfrac{1}{2} \sum \pi_i S^i{}_{i'} (K^{-1})^{i'j'} S^j{}_{j'} \pi_j + \tfrac{1}{2} \sum x^i (S^{-1})^{i'}{}_i V_{i'j'} (S^{-1})^{j'}{}_j x^j \tag{14}$$

(d) We now compute the equal-time commutation relations for the xs and πs from those for the qs and ps, and show they agree with the standard ETCCR. At equal time, the qs commute with each other, as do the ps, so the same is true of the xs and πs. The remaining ETCCR for the new coordinates are

$$\begin{aligned}[x^j(t), \pi_k(t)] &= \sum_{j'k'} S^j{}_{j'} (S^{-1})^{k'}{}_k [q^{j'}(t), p_{k'}(t)] \\ &= i\hbar \sum_{j'k'} S^j{}_{j'} (S^{-1})^{k'}{}_k \delta^{j'}{}_{k'} \\ &= i\hbar \sum_{j'} S^j{}_{j'} (S^{-1})^{j'}{}_k \\ &= i\hbar \delta^j{}_k. \end{aligned} \tag{15}$$

So the ETCCR are preserved.

(e) Possible ideas for further consistency conditions

(i) Show that the equations of motion in the old coordinates, as given by $\mathrm{d}q/\mathrm{d}t = \partial H/\partial p$, etc, when the variable change is applied, give the corresponding equation in the new coordinates.

(ii) Similarly changing variables in the Euler-Lagrange equations

$$0 = \frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}^i} \tag{16}$$

gives the Euler-Lagrange equations in the new coordinates.

(iii) $H(x, \pi)$ was obtained from $L(x, \dot{x})$ by the usual procedure. Verify that it is obtain from the Hamiltonian $H(q, p)$ in the old coordinates by a change of variable.

In each case we have two ways of obtaining an object in the new coordinates:

- From L apply the standard prescriptions in the old coordinates and then change variables.
- Change variables in L and then apply the standard prescription in the new coordinates.

The question being addressed is whether the two orders of applying the change of variable and applying the prescription agree.