A real symmetric matrix can be diagonalised by applying a real orthogonal matrix. Thus

$$K = A^T D A$$

where D is diagonal: $D_{ij} = \delta_{ij}\lambda_j$ & $A^T = A^{-1}$

$$K_{ij} = \sum_{i'} (A)^{i'}{}_{i}\, \lambda_{i'}\, A^{i'}{}_{j}$$

Define $x^i = A^i{}_{i'} q^{i'}$; so $q = A^{-1}x$. Then

$$L = \frac{1}{2} \dot{x}^T (A^{-1})^T A^T D A A^{-1} \dot{x} - \frac{1}{2} x^T A^{+1} V A^{-1} x$$

$$= \frac{1}{2} \dot{x}^T D \dot{x} - \frac{1}{2} x^T A V A^T x$$

$$= \frac{1}{2} \sum_i \dot{x}^{i2} \lambda_i - \frac{1}{2} \sum_{ij} x^i \hat{V}_{ij} x^j,$$

where $\hat{V}_{ij} = \sum A^i{}_{i'} V_{i'j'} A^j{}_{j'}$.

K_{ij} is a positive matrix so all its eigenvalues are positive $\lambda_i > 0$. [V is a positive matrix means that $y^T K y > 0$ for all nonzero vectors y.]

Now let $y^i = \sqrt{\lambda_i}\, x^i$, so $x^i = \frac{1}{\sqrt{\lambda_i}} y^i$

$$L = \frac{1}{2}\sum_i \dot{y}^{i\,2} - \frac{1}{2}\sum_{ij} y^i W_{ij} y^j,$$

where $W_{ij} = \frac{1}{\sqrt{\lambda_i \lambda_j}} \hat{V}_{ij} = \sum_{i'j'} \frac{1}{\sqrt{\lambda_i \lambda_j}} A^{i'}{}_i V_{i'j'} A^{j'}{}_j$.

Now diagonalize W, also a positive matrix, by an orthonormal transformation B

$$W_{ij} = \sum_\alpha B^\alpha{}_i\, \omega_\alpha^2\, B^\alpha{}_j,$$

where we write the positive eigenvalues as ω_α's.

Let $Q^\alpha = \sum_j B^\alpha_j y^j$; $\quad Q = By$

$$y = B^{-1} Q = B^T Q$$

$$\dot{y}^T \dot{y} = \dot{Q}^T B B^{-1} \dot{Q} = \dot{Q}^T \dot{Q}^T$$

$$\therefore L = \sum_\alpha \left(\frac{1}{2}(\dot{Q}^\alpha)^2 - \frac{1}{2}\omega_\alpha^2 (Q^\alpha)^2 \right).$$

$Q^\alpha = \sum B^\alpha{}_j \sqrt{\lambda_j}\, A^j{}_k q^k$, a linear transformation

$= \sum M^\alpha{}_k q^k$ where $M^\alpha_k = \sum_j B^\alpha{}_j \sqrt{\lambda_j}\, A^j{}_k$.

(b) We have a sum of N indt SHOs, so general solution is

$$Q^\alpha(t) = \left(a_\alpha e^{-i\omega_\alpha t} + a_\alpha^+ e^{i\omega_\alpha t}\right) c_\alpha,$$

where c_α is a normalization coefficient we will choose & a_α & a_α are the coefficients in the solution.

$$q^j(t) = \sum_\alpha c_\alpha \left(a_\alpha (M^{-1})^j{}_\alpha e^{-i\omega_\alpha t} + a_\alpha^+ (M^{-1})^j{}_\alpha e^{i\omega_\alpha t}\right)$$

Normal mode solutions — one fixed α

$$(M^{-1})^j{}_\alpha e^{-i\omega_\alpha t} \quad \text{\& conjugate}$$

There are N cases, obviously one for each α.

Orthogonality try

$$\sum_{ij} (M^{-1})^i{}_\alpha K_{ij} (M^{-1})^j{}_\beta = \sum_{ijkl} (A^{-1})^i{}_k \frac{1}{\sqrt{\lambda_k}} (B^{-1})^k{}_\alpha \, K_{ij}$$

$$(A^{-1})^j{}_l \frac{1}{\sqrt{\lambda_l}} (B^{-1})^l{}_\beta$$

Now $\sum_{ij} (A^{-1})^i{}_k K_{ij} (A^{-1})^j{}_l = \left((A^{-1})^T K A^{-1}\right)_{kl}$

$$= (D)_{kl}$$

$$= \delta_{kl} \lambda_k .$$

$$\sum (M^{-1})^i{}_\alpha K_{ij} (M^{-1})^j{}_\beta = \sum_k (B^{-1})^k{}_\alpha (B^{-1})^\ell{}_\beta$$
$$= \left((B^{-1})^T B^{-1}\right)_{\alpha\beta}$$
$$= (B B^{-1})_{\alpha\beta}$$
$$= \delta_{\alpha\beta}.$$

(d) $P_\alpha = \dfrac{\partial L}{\partial \dot{Q}^\alpha} = \dot{Q}^\alpha$

$$H = \frac{1}{2}\sum_\alpha \left[(P_\alpha)^2 + \frac{1}{2}\omega_\alpha^2 (Q^\alpha)^2\right].$$

Now $Q^\alpha = c_\alpha\left(a_\alpha e^{-i\omega_\alpha t} + a_\alpha^+ e^{i\omega_\alpha t}\right)$

so $P_\alpha = \dot{Q}^\alpha = i c_\alpha \omega_\alpha\left(-a_\alpha e^{-i\omega_\alpha t} + a_\alpha^+ e^{i\omega_\alpha t}\right)$

$$a_\alpha = \frac{Q^\alpha(t) + i P_\alpha(t)/\omega_\alpha}{2c_\alpha}\, e^{i\omega_\alpha t}$$

$$a_\alpha^+ = \frac{Q^\alpha - i P_\alpha/\omega_\alpha}{2c_\alpha}\, e^{-i\omega_\alpha t}$$

$$[a_\alpha, a_\beta] = \frac{e^{i(\omega_\alpha+\omega_\beta)t}}{4c_\alpha c_\beta}\Big\{[Q^\alpha(t), Q^\beta(t)] + \frac{i}{\omega_\alpha}[P_\alpha(t), Q^\beta(t)] + \frac{i}{\omega_\beta}[Q^\alpha(t), P_\beta(t)] - \frac{1}{\omega_\alpha\omega_\beta}[P_\alpha(t), P_\beta(t)]\Big\}$$

$$= \frac{e^{i(\omega_\alpha+\omega_\beta)t}}{4c_\alpha c_\beta}\, i\hbar\left(\frac{-i}{\omega_\alpha}\delta_{\alpha\beta} + \frac{i}{\omega_\beta}\delta_{\alpha\beta}\right) = 0$$

$$[a_\alpha^+, a_\beta^+] = 0.$$

$$[a_\alpha, a_\beta^+] = \frac{e^{i(\omega_\alpha-\omega_\beta)t}}{4c_\alpha c_\beta}\Big\{[Q^\alpha(t), Q^\beta(t)] - \frac{i}{\omega_\beta}[Q^\alpha(t), P_\beta(t)] + \frac{i}{\omega_\alpha}[P_\alpha(t), Q^\beta(t)] + \frac{1}{\omega_\alpha\omega_\beta}[P_\alpha(t), P_\beta(t)]\Big\}$$

$$= \frac{e^{i(\omega_\alpha-\omega_\beta)t}}{4c_\alpha c_\beta}\, i\hbar\delta_{\alpha\beta}\left(-\frac{i}{\omega_\beta} - \frac{i}{\omega_\alpha}\right)$$

$$= \frac{\hbar}{2c_\alpha^2\omega_\alpha}\delta_{\alpha\beta}.$$

choose $c_\alpha = \sqrt{\frac{\hbar}{2\omega_\alpha}}$ to get

$$[a_\alpha, a_\beta] = [a_\alpha^+, a_\beta^+] = 0, \quad [a_\alpha, a_\beta^+] = \delta_{\alpha\beta}.$$

(e) $$H = \frac{1}{2}\sum_\alpha c_\alpha^2\Big\{-\omega_\alpha^2\left(a_\alpha^2 e^{-2i\omega_\alpha t} - a_\alpha a_\alpha^+ - a_\alpha^+ a_\alpha + (a_\alpha^+)^2 e^{2i\omega_\alpha t}\right) + \omega_\alpha^2\left(a_\alpha^2 e^{-2i\omega_\alpha t} + a_\alpha a_\alpha^+ + a_\alpha^+ a_\alpha + (a_\alpha^+)^2 e^{2i\omega_\alpha t}\right)\Big\}$$

$$= \sum_\alpha c_\alpha^2\omega_\alpha^2\left(a_\alpha a_\alpha^+ + a_\alpha^+ a_\alpha\right)$$

$$= \sum_\alpha \frac{\hbar\omega_\alpha}{2}\left(a_\alpha a_\alpha^+ + a_\alpha^+ a_\alpha\right)$$

$$= \sum_\alpha \hbar\omega_\alpha\left(a_\alpha^+ a_\alpha + \frac{1}{2}\right)$$