$$\phi = \sum_k \left( a_k e^{-i|k|t + i\vec{k}\cdot\vec{x}} + a_k^+ e^{i|k|t - i\vec{k}\cdot\vec{x}} \right)$$

$$\sum_k \to \int \frac{d^3\vec{k}}{(2\pi)^3 |\vec{k}|}$$

Define $$\phi_a(t) = \frac{\int d^3\vec{x}\, e^{-x^2/a^2} \phi(\vec{x}, t)}{\int d^3\vec{x}\, e^{-x^2/a^2}}.$$

Now $$\int d^3\vec{x}\, e^{-x^2/a^2} e^{i\vec{k}\cdot\vec{x}} = \int d^3\vec{x}\, e^{-\frac{1}{a^2}\left(\vec{x} - \frac{i\vec{k}a^2}{2}\right)^2} e^{-k^2a^2/4}$$

$$= \int d^3\vec{x}\, e^{-x^2/a^2} e^{-k^2a^2/4}$$

by shift of variable, which is still valid if it's a complex shift

$$\therefore \phi_a(t) = \sum_k \left( a_k e^{-i|k|t} + a_k^+ e^{i|k|t} \right) e^{-\vec{k}^2a^2/4}$$

$$\therefore \langle 0 | \phi_a(t)^2 | 0 \rangle = \sum_{k\ell} \langle 0 | \left( a_k e^{-i|k|t} + a_k^+ e^{+i|k|t} \right) \cdot \left( a_\ell e^{-i|\ell|t} + a_\ell^+ e^{i|\ell|t} \right) | 0 \rangle \times e^{-(\vec{k}^2 + \vec{\ell}^2)a^2/4}$$

Now $\langle 0 | a^+ = 0$ & $a|0\rangle = 0$, so the matrix element is

$$\langle 0 | a_k e^{-i|k|t} a_\ell^+ e^{i|\ell|t} | 0 \rangle = e^{i(|k| - |\ell|)t} \langle 0 | (a_\ell^+ a_k + \delta_{k\ell}) | 0 \rangle$$

$$= \delta_{k\ell}.$$

$$\langle 0|\phi_a(t)^2|0\rangle = \sum_{\vec{k}} e^{-\vec{k}^2 a^2/2}$$

$$= \int \frac{d^3k}{(2\pi)^3 2|k|} e^{-\vec{k}^2 a^2/2}$$

$$= \frac{1}{(2\pi)^3 \cdot 2} \times 4\pi \int_0^\infty dk\, k^2 \frac{1}{k} e^{-k^2 a^2/2}$$

$$= \frac{1}{4\pi^2}\left[e^{-k^2a^2/2}\left(\frac{-1}{a^2}\right)\right]_{k=0}^{\infty}$$

$$= \frac{1}{4\pi^2 a^2}.$$

Natural units:

H includes term $\int d^3x \frac{1}{2}(\vec{\nabla}\phi)^2$, i.e. dimension $L \cdot [\phi]^2$

$[\phi] = \sqrt{H/L} = \sqrt{(\text{Energy})^2} = \text{energy}$

Potential: $1 \frac{\text{GeV}}{\sqrt{\varepsilon_0 \hbar c}} = 3 \times 10^8 V$,

Now $L$ is $\frac{\hbar c}{\text{GeV}}$

So $a_{\text{SI units}} = a_{\text{nat units}} \hbar c$

$$\text{RMS } \phi_{\text{SI units}} = \text{RMS } \phi_{\text{natural}} \times \frac{1}{\sqrt{\varepsilon_0 \hbar c}}$$

$$= \frac{1}{2\pi a_{nat}} \frac{1}{\sqrt{\varepsilon_0 \hbar c}}$$

$$= \frac{1}{2\pi a_{SI}} \hbar c \frac{1}{\sqrt{\varepsilon_0 \hbar c}} = \frac{1}{2\pi a_{SI}} \sqrt{\frac{\hbar c}{\varepsilon_0}}$$

$$= \frac{1}{a_{SI}} \times 9.5 \times 10^{-9}\ V\text{-m}.$$

| | $a$ | |
|---|---|---|
| | 1 m | $10^{-8}$ V |
| atomic | $10^{-10}$ m | 100 V |
| nuclear | $10^{-15}$ m | 10 MV |

Potential energy in atom: $\sim 10$ eV, so potentials (for 1 electron) are 10 V up

Potential energy for nuclei is MeV to 10s of MeV.

In both cases the variations in potential could matter.