2.(a) One particle state corresponding to $\tilde{f}(\vec{k})$

$$|f\rangle = A \sum_{\vec{k}} |\vec{k}\rangle \tilde{f}(\vec{k})$$

To make this normalized we need

$$1 = \langle f|f\rangle = |A|^2 \sum_{k\ell} \tilde{f}(k)^* \langle \vec{k}|\vec{\ell}\rangle \tilde{f}(\vec{\ell})$$

$$= |AC|^2 \sum_{\vec{k}} e^{-2(\vec{k}-\vec{\ell}_1)^2/\Delta^2}$$

$$= |AC|^2 \int \frac{d^3\vec{k}}{(2\pi)^3 2E_k} e^{-2(\vec{k}-\vec{\ell}_1)^2/\Delta^2}$$

We'll choose

$$C = \left[\int \frac{d^3\vec{k}}{(2\pi)^3 2E_k} e^{-2(\vec{k}-\vec{\ell}_1)^2/\Delta^2} \right]^{-1/2},$$

so that we can set $A=1$. Then a normalized one-particle state is

$$|f\rangle = \sum_{\vec{k}} a^{+}_{\vec{k}}|0\rangle \tilde{f}(\vec{k})$$

(b) Similarly set

$$D = \left[\int \frac{d^3\vec{k}}{(2\pi)^3 2E_k} e^{-2(\vec{k}-\vec{\ell}_2)^2/\Delta^2} \right]^{-1/2}$$

so that $|g\rangle = \sum_{\vec{k}} a_k^+ |0\rangle \tilde{g}(\vec{k})$ is normalized.

Note that

$$\langle f|g\rangle = \sum_{\vec{k},\vec{\ell}} \tilde{f}(\vec{k})^* \langle \vec{k}|\vec{\ell}\rangle \tilde{g}(\vec{\ell})$$

$$= \int \frac{d^3\vec{k}}{(2\pi)^3 2E_k} C D e^{-(\vec{k}-\vec{\ell}_1)^2/\Delta^2} e^{-(\vec{k}-\vec{\ell}_2)^2/\Delta^2}$$

$\tilde{f}(\vec{k})$ is peaked when $\vec{k}$ is within Δ of $\vec{\ell}_1$ & then g is small, since $|\ell_1 - \ell_2| \gg \Delta$. Similarly $\tilde{f}$ is small at the peak of $\tilde{g}$. Hence the integral is small and $|f\rangle$ & $|g\rangle$ are almost non-overlapping.

Hence a one-particle state with 10% probability of being in state $|f\rangle$ and 90% in $|g\rangle$ is (to a good accuracy

$$\sqrt{1/10}\, |f\rangle + \sqrt{9/10}\, |g\rangle$$

$$= \frac{1}{\sqrt{10}} \sum_{\vec{k}} a_k^\dagger |0\rangle \left[\tilde{f}(\vec{k}) + 3\tilde{g}(\vec{k}) \right].$$

(c) Let a 2 particle state be

$$|fg\rangle = \alpha \sum_{\vec{k},\vec{l}} a_k^+ a_l^+ |0\rangle \tilde{f}(\vec{k}) \tilde{g}(\vec{l})$$

To normalize it

$$1 = \langle fg|fg\rangle = |\alpha|^2 \sum_{k_1 k_2 l_1 l_2} \tilde{f}(\vec{k}_1)^* \tilde{g}(\vec{l}_1)^* \tilde{f}(\vec{k}_2) \tilde{g}(\vec{l}_2) \times \langle 0| a_{l_1} a_{k_1} a_{k_2}^+ a_{l_2}^+ |0\rangle$$

In the matrix element commute the a_{k_1} & a_{l_1} to the right to get

$$\langle 0| a_{l_1} a_{k_1} a_{k_2}^+ a_{l_2}^+ |0\rangle = \langle 0| a_{l_1} a_{k_2}^+ a_{k_1} a_{l_2}^+ |0\rangle + \delta_{k_1 k_2} \langle 0| a_{l_1} a_{l_2}^+ |0\rangle$$

$$= \langle 0| a_{l} a_{k_2}^+ a_{l_2}^+ a_{k_1} |0\rangle + \delta_{k_1 l_2} \langle 0| a_{l_1} a_{k_2}^+ |0\rangle$$

$$+ \delta_{k_1 k_2} \left[\delta_{l_1 l_2} + \langle 0| a_{l_2}^+ a_{l_1} |0\rangle \right]$$

$$= 0 + \delta_{k_1 l_2} \left[\langle 0| a_{k_2}^+ a_{l_1} |0\rangle + \delta_{k_2 l_1} \right] + \delta_{k_1 k_2} \delta_{l_1 l_2}$$

$$= \delta_{k_1 l_2} \delta_{k_2 l_1} + \delta_{k_1 k_2} \delta_{l_1 l_2} .$$

$$\therefore 1 = |\alpha|^2 \sum_{l_1 l_2} \left[\tilde{f}(\vec{l}_2)^* \tilde{g}(\vec{l}_1)^* \tilde{f}(\vec{l}_1) \tilde{g}(\vec{l}_2) + \tilde{f}(\vec{l}_2)^* \tilde{g}(\vec{l}_1)^* \tilde{f}(\vec{l}_2) \tilde{g}(\vec{l}_1) \right]$$

$$= |\alpha|^2 \left[|\langle f|g\rangle|^2 + \langle f|f\rangle \langle g|g\rangle \right]$$

$$\simeq |\alpha|^2 .$$

$\therefore$ we can set $\alpha = 1$ to give $|fg\rangle = \sum_{k,l} a_k^+ a_l^+ |0\rangle \tilde{f}(\vec{k}) \tilde{g}(\vec{l})$.

(d) Let

$$|ff\rangle = \beta \sum_{\vec{k},\vec{l}} a^\dagger_{k} a^\dagger_{l} |0\rangle \, \tilde{f}(\vec{k}) \, \tilde{f}(\vec{l})$$

$$1 = \langle ff|ff\rangle = |\beta|^2 \sum_{k_1,k_2,l_1,l_2} \tilde{f}(\vec{k}_1)^* \tilde{f}(\vec{l}_1)^* \tilde{f}(\vec{k}_2) \tilde{f}(\vec{l}_2) \times \langle 0 | a_{l_1} a_{k_1} a^\dagger_{k_2} a^\dagger_{l_2} |0\rangle$$

$$= |\beta|^2 \sum_{l_1,l_2} \left[\tilde{f}(\vec{l}_2)^* \tilde{f}(\vec{l}_1)^* \tilde{f}(\vec{l}_1) \tilde{f}(\vec{l}_2) + \tilde{f}(\vec{l}_2)^* \tilde{f}(\vec{l}_1)^* \tilde{f}(\vec{l}_2) \tilde{f}(\vec{l}_1) \right]$$

$$= |\beta|^2 \times 2 \times \langle f|f\rangle^2$$

$$= 2|\beta|^2$$

$\therefore$ choose $\beta = 1/\sqrt{2}$ to get

$$|ff\rangle = \frac{1}{\sqrt{2}} \sum_{\vec{k},\vec{l}} a^\dagger_{\vec{k}} a^\dagger_{\vec{l}} |0\rangle \, \tilde{f}(\vec{k}) \, \tilde{f}(\vec{l}).$$

(e) All the conditions can be met if C & D are multiplied by (possibly different) phases, i.e. by replacing C by $Ce^{i\gamma}$ & D by $De^{i\delta}$, where γ & δ are any real numbers. This gives an overall phase arbitrariness in all parts and an arbitrary relative phase between the 2 components in (a).

Not essential to solution

Approximate normalization factor if Δ small

$$1 = |C|^2 \int \frac{d^3\vec{k}}{(2\pi)^3 2E_k} \, e^{-(\vec{k}-\vec{\ell}_1)^2 \frac{2}{\Delta^2}}$$

$$\simeq \frac{|C|^2}{16\pi^3 E_{\ell_1}} \int d^3\vec{k} \, e^{-(\vec{k}-\vec{\ell}_1)^2 \frac{2}{\Delta^2}}$$

where the variation of E_k with k is neglected

$$= \frac{|C|^2}{16\pi^3 E_{\ell_1}} \int d^3\vec{k} \, e^{-\vec{k}^2 \times 2/\Delta^2}$$

$$= \frac{|C|^2}{16\pi^3 E_{\ell_1}} \left(\frac{\pi \Delta^2}{2}\right)^{3/2}$$

$$\therefore |C| = 4 (2\pi)^{3/2} \frac{E_{\ell_1}}{\Delta^{3/2}}$$