$$L = \frac{1}{2}(q_1\dot{q}_2 - q_2\dot{q}_1) - f(q_1, q_2).$$

This is antisymmetric in the $q_1\dot{q}_2$ & $q_2\dot{q}_1$ terms.
According to the Faddeev-Jackiw rules, set

$$p_1 = \frac{\partial L}{\partial \dot{q}_1} = -q_2/2, \quad 2[q_1, p_1] = i\hbar, \text{ i.e. } [q_2, q_1] = i\hbar$$

$$p_2 = \frac{\partial L}{\partial \dot{q}_2} = q_1/2, \quad 2[q_2, p_2] = i\hbar, \text{ i.e. } [q_2, q_1] = i\hbar$$

$$H = \sum p_i\dot{q}_i - L = -\frac{q_2}{2}\dot{q}_1 + \frac{q_1}{2}\dot{q}_2 - L = f(q_1, q_2)$$

Euler-Lagrange eqs

$$0 = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} + \frac{\partial L}{\partial q_1} = -\frac{d}{dt}\left(-\frac{q_2}{2}\right) + \frac{1}{2}\dot{q}_2 - \frac{\partial f}{\partial q_1}$$

$$= \dot{q}_2 - \frac{\partial f}{\partial q_1})$$

so $\dot{q}_2 = \frac{\partial f}{\partial q_1}$

$$0 = -\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} + \frac{\partial L}{\partial q_2} = -\dot{q}_1 - \frac{\partial f}{\partial q_2}.$$

$$\dot{q}_1 = -\frac{\partial f}{\partial q_2}.$$

From Heisenberg

$$\dot{q}_1 = -\frac{i}{\hbar}[q_1, H] = -\frac{i}{\hbar}[q_1, q_2]\frac{\partial H}{\partial q_2} = -1 \times \frac{\partial f}{\partial q_2} = -\frac{\partial f}{\partial q_2}$$

(except for operator ordering issues in f)

$$\dot{q}_2 = -\frac{i}{\hbar}[q_2, H] = -\frac{i}{\hbar}[q_2, q_1]\frac{\partial H}{\partial q_1}$$
$$= \frac{\partial f}{\partial q_1}$$

These agree with E-L eq.

Now set $f = \frac{\omega}{2}(q_1^2 + q_2^2)$.

$$\dot{q}_1 = -\frac{\partial f}{\partial q_2} = -\omega q_2$$

$$\dot{q}_2 = \frac{\partial f}{\partial q_1} = \omega q_1$$

$$\ddot{q}_1 = -\omega^2 q_1$$

So general solution is $q_1 = A e^{-i\omega t} + A^\dagger e^{i\omega t}$
for some operator A

$$q_2 = -\frac{1}{\omega}\dot{q}_1 = iAe^{-i\omega t} - iA^\dagger e^{i\omega t}$$

$$A = \frac{1}{2}e^{i\omega t}(q_1 - iq_2)$$
$$A^\dagger = \frac{1}{2}e^{-i\omega t}(q_1 + iq_2)$$
$$[A, A^\dagger] = \frac{1}{4}[q_1 - iq_2, q_1 + iq_2] = \frac{\hbar}{2}$$

This is proportional to SHO formulae. To get standard normalization, let $a = A\sqrt{\frac{2}{\hbar}}$, so $[a, a^\dagger] = 1$

$$q_1 = \sqrt{\frac{\hbar}{2}}\left(a e^{-i\omega t} + a^\dagger e^{i\omega t}\right)$$

$$q_2 = \sqrt{\tfrac{\hbar}{2}}\left(i a e^{-i\omega t} - i a^\dagger e^{i\omega t}\right).$$

$$H = \frac{\omega\hbar}{4}\left[\left(a e^{-i\omega t} + a^\dagger e^{i\omega t}\right)^2 + i^2\left(a e^{-i\omega t} - a^\dagger e^{i\omega t}\right)^2\right]$$

$$= \frac{\hbar\omega}{2}\left(a a^\dagger + a^\dagger a\right)$$

$$= \hbar\omega\left(a^\dagger a + \tfrac{1}{2}\right).$$

This is just SHO, just by recognition of formulae for H & $[a, a^\dagger]$ $\longrightarrow$ same H, with q_1 = coordinate & q_2 = standard momentum. (Up to normalization)