**Solution to: problem set 4, question 3**

We are given

$$L = \frac{i}{2}\left(\psi^\dagger \dot{\psi} - \dot{\psi}^\dagger \psi\right) - \omega \psi^\dagger \psi. \tag{1}$$

The Euler-Lagrange equations are

$$0 \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\psi}} - \frac{\partial L}{\partial \psi} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{i}{2}\psi^\dagger\right) + \frac{i}{2}\dot{\psi}^\dagger + \omega\psi^\dagger \;=\; i\dot{\psi}^\dagger + \omega\psi^\dagger, \tag{2}$$

and exactly similarly

$$0 \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{\psi}^\dagger} - \frac{\partial L}{\partial \psi^\dagger} \;=\; -\,i\dot{\psi} + \omega\psi. \tag{3}$$

Hence

$$\dot{\psi} = -i\omega\psi, \qquad \dot{\psi}^\dagger = i\omega\psi^\dagger. \tag{4}$$

The Lagrangian has the form

$$L = \frac{1}{2}\sum_{j,k} q^j f_{jk} \dot{q}^k - F(\underline{q}). \tag{5}$$

to which the Floreanini-Faddeev-Jackiw rules apply, with

$$f_{\psi^\dagger\psi} = i, \quad f_{\psi\psi^\dagger} = -i, \quad f_{\psi\psi} = f_{\psi^\dagger\psi^\dagger} = 0. \tag{6}$$

This matrix is its own inverse, so that the rule

$$[q^j(t), q^k(t)] = i\hbar (f^{-1})^{jk}. \tag{7}$$

gives

$$[\psi(t), \psi^\dagger(t)] = i\hbar(-i) = \hbar. \tag{8}$$

The Hamiltonian is

$$H(\psi,\psi^\dagger) = \frac{\partial L}{\partial \dot{\psi}^\dagger}\dot{\psi}^\dagger + \frac{\partial L}{\partial \dot{\psi}}\dot{\psi} - L = \omega\psi^\dagger\psi. \tag{9}$$

(Note that a quantity multiplying the derivative with respect to that quantity is defined to put the quantity back where the derivative took it out. Thus the ordering of the first two terms is the same as in $L$.)

The general solution of the equations of motion is

$$\psi(t) = a\sqrt{\hbar}e^{-i\omega t}, \qquad \psi^\dagger(t) = a^\dagger\sqrt{\hbar}e^{i\omega t}, \tag{10}$$

where the operator $a$ is time-independent, and the normalization factor $\sqrt{\hbar}$ is adjusted so that the commutator of $a$ and $a^\dagger$ is

$$[a, a^\dagger] = 1. \tag{11}$$

The Hamiltonian is then

$$H(\psi, \psi^\dagger) = \hbar\omega a^\dagger a. \tag{12}$$

The last two equations are the same as the standard SHO, with the energy shifted by $\hbar\omega/2$.

To convert the system to the previous problem, write $\psi$ and $\psi^\dagger$ in terms of real and imaginary parts:

$$\psi(t) = \frac{1}{\sqrt{2}}(q_1(t) - iq_2(t)), \tag{13}$$

$$\psi^\dagger(t) = \frac{1}{\sqrt{2}}(q_1(t) + iq_2(t)). \tag{14}$$

Then the Lagrangian becomes

$$\begin{aligned} L &= \frac{i}{4}\left[(q_1(t) + iq_2(t))(\dot{q}_1(t) - i\dot{q}_2(t)) - (\dot{q}_1(t) + i\dot{q}_2(t))(q_1(t) - iq_2(t))\right] \\ &\quad - \omega(q_1(t) + iq_2(t))(q_1(t) - iq_2(t)) \\ &= \frac{i}{4}\left[q_1(t)\dot{q}_1(t) - \dot{q}_1(t)q_1(t) + q_2(t)\dot{q}_2(t) - \dot{q}_2(t)q_2(t)\right] \\ &\quad + \frac{i}{4}\left[-iq_1(t)\dot{q}_2(t) + iq_2(t)\dot{q}_1(t) + i\dot{q}_1(t)q_2(t) - i\dot{q}_2(t)q_1(t)\right] \\ &\quad - \omega(q_1(t)^2 + q_2(t)^2 - iq_1(t)q_2(t) + iq_2(t)q_1(t)) \\ &= \frac{1}{2}(q_1(t)\dot{q}_2(t) - q_2(t)\dot{q}_1(t)) - \omega(q_1(t)^2 + q_2(t)^2) \end{aligned} \tag{15}$$

*up to operator ordering issues*. Changing the operator ordering at most adds a numerical constant, so it only changes the Hamiltonian by a constant energy. Thus, after change of variables, the system has the same Lagrangian as the one in the previous problem, up to a constant.