Solution to: problem set 5, question 2

In this problem I will give longer explanations than are necessary for your solutions to the problem, in order to cover some tricky conceptual issues that sometimes get bothersome.

Solution to problem

Consider a Lagrangian $L(q, \dot{q})$, where q denotes a list $(q^1, q^2, \ldots)$ of generalized coordinates. We consider invariance under a group of transformations of the coordinates with parameters ω_α:

$$q \mapsto q_{\text{new}}(q, \omega). \tag{1}$$

Let the linearized transformations be

$$q^j \mapsto q^j + \sum_\alpha \omega_\alpha D^\alpha q^j + O(\omega^2), \tag{2}$$

where we define

$$D^\alpha q^j = \left. \frac{\mathrm{d}q^j_{\text{new}}(q, \omega)}{\mathrm{d}\omega_\alpha} \right|_{\text{Given } q, \text{ at } \omega = 0}. \tag{3}$$

The variation of the Lagrangian under the transformation is $L \mapsto L + \sum_\alpha \omega_\alpha D^\alpha L + O(\omega^2)$, where

$$\begin{aligned} D^\alpha L &= \left. \frac{\mathrm{d}L(q_{\text{new}}, \dot{q}_{\text{new}})}{\mathrm{d}\omega_\alpha} \right|_{\text{Given } q,\, \dot{q}, \text{ at } \omega = 0} \\ &= \frac{\partial L}{\partial q^j} D^\alpha q^j + \frac{\partial L}{\partial \dot{q}^j} \frac{\mathrm{d}D^\alpha q^j}{\mathrm{d}t}, \qquad \text{(general)}, \end{aligned} \tag{4}$$

with the summation condition understood. The above formula applies in all cases.

When there is a symmetry (at least of the kind we treat), the linearized transformation of L is a total time derivative

$$D^\alpha L = \frac{\mathrm{d}f^\alpha(q, \dot{q}, t)}{\mathrm{d}t} \qquad \text{(symmetry)}, \tag{5}$$

where f^α is a function of the coordinates, velocities, and time *alone*, and this equation applies *without* the use of the Euler-Lagrange equations of motion. A simple invariance of L, like rotation invariance, would give $f^\alpha = 0$.

Define the Noether charges, one for each generator of the symmetry group, by

$$Q^\alpha = \frac{\partial L}{\partial \dot{q}^j} D^\alpha q^j - f^\alpha. \tag{6}$$

Then

$$\begin{aligned}
\frac{\mathrm{d}Q^\alpha}{\mathrm{d}t} &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^j}\right) D^\alpha q^j + \frac{\partial L}{\partial \dot{q}^j}\frac{\mathrm{d}}{\mathrm{d}t}(D^\alpha q^j) - \frac{\mathrm{d}f^\alpha}{\mathrm{d}t} \\
&= \frac{\partial L}{\partial q^j} D^\alpha q^j + \frac{\partial L}{\partial \dot{q}^j} D^\alpha \dot{q}^j - \frac{\mathrm{d}f^\alpha}{\mathrm{d}t} && \text{by the equations of motion} \\
&= D^\alpha L - \frac{\mathrm{d}f^\alpha}{\mathrm{d}t} \\
&= 0 && \text{by (5).} \qquad (7)
\end{aligned}$$

Non-trivial example (not required for solution)

Consider the usual Lagrangian for a particle with a potential in one dimension:

$$L = \frac{m}{2}\dot{q}(t)^2 - V(q(t)). \qquad (8)$$

The transformations we consider are scaling of time, with a particular simultaneous scaling of the coordinate:

$$q_{\text{new}}(t) = e^{-\frac{1}{2}\omega} q(e^\omega t). \qquad (9)$$

Here ω is the parameter of the transformations. The corresponding velocity is

$$\begin{aligned}
\frac{\mathrm{d}q_{\text{new}}(t)}{\mathrm{d}t} &= \frac{\mathrm{d}\left[e^{-\frac{1}{2}\omega} q(e^\omega t)\right]}{\mathrm{d}t} \\
&= e^{\frac{1}{2}\omega}\dot{q}(e^\omega t), \qquad (10)
\end{aligned}$$

where $\dot{q}(e^\omega t)$ means

$$\dot{q}(e^\omega t) = \frac{\mathrm{d}\, q(e^\omega t)}{\mathrm{d}\, e^\omega t}. \qquad (11)$$

The linearized transformations use

$$\begin{aligned}
Dq &= \left.\frac{\mathrm{d}\left[e^{-\frac{1}{2}\omega} q(e^\omega t)\right]}{\mathrm{d}\omega}\right|_{\text{Given } q, \text{ at } \omega = 0} \\
&= t\dot{q}(t) - \frac{1}{2}q(t), \qquad (12) \\
D\dot{q} &= \left.\frac{\mathrm{d}\left[e^{\frac{1}{2}\omega} \dot{q}(e^\omega t)\right]}{\mathrm{d}\omega}\right|_{\text{Given } q, \text{ at } \omega = 0} \\
&= t\ddot{q}(t) + \frac{1}{2}\dot{q}(t) \\
&= \frac{\mathrm{d}Dq}{\mathrm{d}t}. \qquad (13)
\end{aligned}$$

The transformed action is

$$
\begin{aligned}
S[q_{\rm new}] &= \int L(q_{\rm new}(t), \dot{q}_{\rm new}(t))\, {\rm d}t \\
&= \int \left[e^{\omega} \frac{m}{2} \dot{q}(e^{\omega} t)^2 - V(e^{-\frac{1}{2}\omega} q(e^{\omega} t)) \right] {\rm d}t \\
&= \int \left[\frac{m}{2} \dot{q}(t')^2 - e^{-\omega} V(e^{-\frac{1}{2}\omega} q(t')) \right] {\rm d}t',
\end{aligned}
\tag{14}
$$

by the change of variable $t' = e^{\omega} t$. An invariance means that $S[q_{\rm new}] = S[q]$ for general q. The kinetic energy term is always invariant, but the potential energy is invariant only if it is proportional to $1/q^2$:

$$
V(q) = \frac{\lambda}{q^2}. \tag{15}
$$

This might seem a strange potential, but recall that it occurs when one treats the radial coordinate of the motion of a particle in three dimensions with a definite angular momentum. (E.g., see the radial equation for the wave function of an energy eigenstate in a central potential in ordinary quantum mechanics.)

The linearized transformation of L is

$$
\begin{aligned}
DL &= \frac{\partial L}{\partial \dot{q}} D\dot{q} + \frac{\partial L}{\partial q} Dq \\
&= m\dot{q} \left(t\ddot{q} + \frac{1}{2}\dot{q} \right) - \frac{{\rm d}V}{{\rm d}q} \left(t\dot{q} - \frac{1}{2} q \right).
\end{aligned}
\tag{16}
$$

For the case of the potential that gives a symmetry, (15), we have

$$
\begin{aligned}
DL &= m\dot{q} \left(t\ddot{q} + \frac{1}{2}\dot{q} \right) + \frac{2\lambda}{q^3} \left(t\dot{q} - \frac{1}{2} q \right) \\
&= \frac{\rm d}{{\rm d}t} \left(\frac{m}{2} t\dot{q}^2 - \frac{\lambda t}{q^2} \right).
\end{aligned}
\tag{17}
$$

The Noether charge is therefore

$$
\begin{aligned}
Q &= \frac{\partial L}{\partial \dot{q}} Dq - \frac{m}{2} t\dot{q}^2 + \frac{\lambda t}{q^2} \\
&= \frac{m}{2} t\dot{q}^2 - \frac{m}{2} q\dot{q} + \frac{\lambda t}{q^2}.
\end{aligned}
\tag{18}
$$

It is easily verified from the equations of motion that ${\rm d}Q \,/\, {\rm d}t = 0$, i.e., that Q is a conserved charge.