Rotation of $\vec{x}$ by angle θ about axis $\vec{n}$ transforms $\vec{x}$ to

$$\vec{x}_{\theta\vec{n}} = \underbrace{\vec{n}\,\vec{x}\cdot\vec{n}}_{\substack{\text{component}\\ \text{of } \vec{x} \text{ ll}^{el} \text{ to } \vec{n}}} + \underbrace{(\vec{x} - \vec{n}\,\vec{x}\cdot\vec{n})}_{\substack{\text{component}\\ \perp \text{ to } \vec{n}}} \text{ rotated by } \theta$$

$$= \vec{n}\,\vec{x}\cdot\vec{n} + (\vec{x} - \vec{n}\,\vec{x}\cdot\vec{n})\cos\theta$$
$$+ \vec{n}\times(\vec{x} - \vec{n}\,\vec{x}\cdot\vec{n})\sin\theta$$

vector product

$$= \vec{x}\cos\theta + \vec{n}\,\vec{x}\cdot\vec{n}(1-\cos\theta) + \vec{n}\times\vec{x}\sin\theta,$$

since $\vec{n}\times\vec{n} = 0$.

We use $\vec{\omega} = \theta\vec{n}$ to represent the three parameters of the transformation.

Any function of $\vec{x}^2$ is invariant & similarly for $\dot{\vec{x}}^2$.

We can see this explicitly from the above formula by noting that

$$\left(\vec{x}_{\theta\vec{n}}\right)^2 = \vec{x}^2\cos^2\theta + 2\vec{x}\cdot\vec{n}\,\vec{x}\cdot\vec{n}(1-\cos\theta)\cos\theta$$
$$+(\vec{n}\,\vec{x}\cdot\vec{n})^2(1-\cos\theta)^2 + (\vec{n}\times\vec{x})^2\sin^2\theta$$

(where we used the fact that $\vec{x}$ & $\vec{n}$ are orthogonal to $\vec{n}\times\vec{x}$)

$$= \vec{x}^2\cos^2\theta + \vec{x}\cdot\vec{n}^2(1-\cos^2\theta) + (\vec{n}\times\vec{x})^2\sin^2\theta$$
$$= \vec{x}^2\cos^2\theta + \left[(\vec{x}\cdot\vec{n})^2 + (\vec{n}\times\vec{x})^2\right]\sin^2\theta.$$

Now note that $|\vec{n}\cdot\vec{x}| = |\vec{n}||\vec{x}|\cos\alpha = |\vec{x}|\cos\alpha$

$$|\vec{n}\times\vec{x}| = |\vec{n}||\vec{x}|\sin\alpha = |\vec{x}|\sin\alpha,$$

where α is the angle between $\vec{n}$ & $\vec{x}$.

Hence $|\vec{x}_{\theta\vec{n}}| = |\vec{x}|$.

The linearized part of the transformation with $\vec{\omega} = \theta\vec{n}$ is

$$\vec{x}_{\vec{\omega}} = \vec{x} + \vec{n}\times\vec{x}\,\theta + O(\theta^2)$$
$$= \vec{x} + \vec{\omega}\times\vec{x} + O(\omega^2)$$

Thus $\omega_i D^i x^j = (\vec{\omega}\times\vec{x})^j = \varepsilon_{jik}\,\omega_i x^k$ (with summation convention).

So $D^i x^j = \varepsilon_{jik} x^k$.

The Lagrangian $L = \frac{1}{2}m\dot{\vec{x}}^2 - V(|\vec{x}|)$ is invariant under rotations & we can verify the linearized from $D^i L = 0$ by

$$D^i L = \frac{\partial L}{\partial \dot{x}^j} D^i \dot{x}^j + \frac{\partial L}{\partial|\vec{x}|}\frac{\partial|\vec{x}|}{\partial x^j} D^i x^j$$
$$= \dot{x}^j \varepsilon_{jik}\dot{x}^k - \frac{\partial V}{\partial|\vec{x}|}\frac{x^j}{|\vec{x}|}\varepsilon_{jik} x^k$$
$$= 0 \text{ by antisymmetry of } \varepsilon$$

From problem 4, the Noether charges are

$$Q^i = \frac{\partial L}{\partial \dot{x}^j} D^i x^j$$

$$= m\dot{x}^j \varepsilon_{jik} x^k \qquad R$$

$$= \varepsilon_{ikj} x^k p^j$$

So $\vec{Q} = \vec{x} \times \vec{p}$, which is the angular momentum

(N.B. in QM the operator ordering of x & p doesn't matter:

$$\varepsilon_{ikj} x^k p^j = \varepsilon_{ikj}\left(p^j x^k + [p^j, x^k]\right)$$

$$= \varepsilon_{ikj}\, p^j x^k - i\hbar\, \delta^{jk} \varepsilon_{ikj}$$

$$= \varepsilon_{ikj}\, p^j x^k,$$

because $[p^j, x^k]$ is symmetric under exchange of j & k, but ε_{ikj} is antisymmetric