(a) $$\left[Q_a, \phi_j(t,\vec{y})\right] = \frac{i}{2}\sum_{kk'}(\sigma_a)_{kk'}\int d^3\vec{x}\left\{\left[\phi_k^\dagger(t,\vec{x})\,\pi_{k'}^\dagger(t,\vec{x}),\ \phi_j(t,\vec{y})\right] - \left[\pi_k(t,\vec{x})\,\phi_{k'}(t,\vec{x}),\ \phi_j(t,\vec{y})\right]\right\}$$

$$= -\frac{i}{2}\sum_{kk'}(\sigma_a)_{kk'}\int d^3\vec{x}\left[\pi_k(x)\,\phi_{k'}(x),\ \phi_j(y)\right],$$

since ϕ_j commutes with $\phi_k^\dagger$ & $\pi_{k'}^\dagger$ at equal times

$$= -\frac{i}{2}\sum_{kk'}(\sigma_a)_{kk'}\int d^3\vec{x}\left(\left[\pi_k(x), \phi_j(y)\right]\phi_{k'}(x) + \pi_k(x)\left[\phi_{k'}(x), \phi_j(y)\right]\right)$$

$$= -\frac{i}{2}\sum_{kk'}(\sigma_a)_{kk'}\int d^3\vec{x}\left(-i\,\delta^{(3)}(\vec{x}-\vec{y})\,\delta_{jk}\right)\phi_{k'}(x)$$

$$= -\frac{1}{2}\sum_{k'}(\sigma_a)_{jk'}\,\phi_{k'}(y)$$

$$= -i\,D_a\phi_j(y).$$

Similarly

$$\left[Q_a, \phi_j^\dagger(t,\vec{y})\right] = \frac{i}{2}\sum(\sigma_a)_{kk'}\int d^3\vec{x}\left\{\left[\phi_k^\dagger(t,\vec{x})\,\pi_{k'}^\dagger(t,\vec{x}),\ \phi_j^\dagger(t,\vec{y})\right] - \left[\pi_k(t,\vec{x})\,\phi_{k'}(t,\vec{x}),\ \phi_j^\dagger(t,\vec{y})\right]\right\}$$

$$= \frac{i}{2}\sum(\sigma_a)_{kk'}\int d^3\vec{x}\ \phi_k^\dagger(t,\vec{x})\left[\pi_{k'}^\dagger(t,\vec{x}),\ \phi_j^\dagger(t,\vec{y})\right]$$

$$= \frac{i}{2}\sum_{kk'}(\sigma_a)_{kk'}\int d^3\vec{x}\ \phi_k^\dagger(x)\left(-i\,\delta^{(3)}(\vec{x}-\vec{y})\,\delta_{k'j}\right)$$

$$= \frac{1}{2}\sum_{k}\phi_k^\dagger(\sigma_a)_{kk'}$$

$$= -i\,D_a\phi_j^\dagger$$

Apply $\frac{\partial}{\partial t}$ to get

$$[Q_a, \dot{\phi}_J] = -i D_a \dot{\phi}_J = -\frac{1}{2} \sum_{J'} (\sigma_a)_{JJ'} \dot{\phi}_{J'}$$

& $$[Q_a, \dot{\phi}_J^+] = \frac{1}{2} \sum_{J'} \dot{\phi}_{J'}^+ (\sigma_a)_{J'J}.$$

[You can also compute these by applying ETCCR.]

(b) To obtain $[Q_a, Q_b]$, write Q_b in terms of fields & use part (a) repeatedly:

$$[Q_a, Q_b] = \frac{i}{2} \sum_{JJ'} (\sigma_b)_{JJ'} \int d^3\vec{x} \left\{ [Q_a, \phi_J^+ \dot{\phi}_{J'}] - [Q_a, \dot{\phi}_J^+ \phi_{J'}] \right\}$$

$$= \frac{i}{2} \sum_{JJ'} (\sigma_b)_{JJ'} \int d^3\vec{x} \left\{ [Q_a, \phi_J^+] \dot{\phi}_{J'} + \phi_J^+ [Q_a, \dot{\phi}_{J'}] - [Q_a, \dot{\phi}_J^+] \phi_{J'} - \dot{\phi}_J^+ [Q_a, \phi_{J'}] \right\}$$

$$= \frac{i}{4} \int d^3\vec{x} \left\{ \phi^+ \sigma_a \sigma_b \dot{\phi} - \phi^+ \sigma_b \sigma_a \dot{\phi} - \dot{\phi}^+ \sigma_a \sigma_b \phi + \dot{\phi}^+ \sigma_b \sigma_a \phi \right\}$$

$$= \frac{i}{4} \int d^3\vec{x} \left\{ \phi^+ [\sigma_a, \sigma_b] \dot{\phi} - \dot{\phi}^+ [\sigma_a, \sigma_b] \phi \right\}$$

$$= \frac{i}{4} 2i \sum_c \varepsilon_{abc} \int d^3\vec{x} \left(\phi^+ \sigma_c \dot{\phi} - \dot{\phi}^+ \sigma_c \phi \right)$$

$$= i \sum_c \varepsilon_{abc} Q_c$$