Physics 563 (Quantum Field Theory 1) Fall 2009

Problem set 1

Due Sep. 4, 2009

1. *Consistency of the Heisenberg formulation:* Consider a quantum mechanical system with coordinates q_i and momenta p^i ($i = 1, \ldots, N$), and a Hamiltonian $H(p, q)$. Define the system by requiring the coordinates and momenta obey the standard ETCCR at $t = 0$ and by their time-evolution being governed by the Heisenberg equations of motion.

(The standard ETCCR are $[q_j(t), p^k(t)] = i\hbar\delta_j^k$, $[q_j(t), q_k(t)] = [p^j(t), p^k(t)] = 0$.)

(a) Suppose operators $A(t)$ and $B(t)$ obey the Heisenberg equations of motion. Show that the product $A(t)B(t)$ and any linear combination of $A(t)$ and $B(t)$ also obey the Heisenberg equations.

(b) Deduce that any polynomial in $q(t)$ and $p(t)$ with constant coefficients obeys the Heisenberg equations.

(c) Show that H is time-independent. You may assume that H is a polynomial in the coordinates and momenta with constant (i.e., time-independent) coefficients.

(d) Show that the ETCCR hold for all times t, given that they hold at $t = 0$.

(e) Show that $\mathcal{O}(t) = e^{iHt/\hbar}\mathcal{O}(0)e^{-iHt/\hbar}$, where $\mathcal{O}(t)$ is any polynomial in $q(t)$ and $p(t)$ with constant coefficients.

(f) Comment on whether/how these results show consistency of the formalism.

Hints: (i) Solution of first order ODE is unique given initial conditions. (ii) Or you could write an ansatz for a solution in terms of an exponential of $H(t = 0)$.

2. Generalize the results of problem 1 to obtain the corresponding results the Schrödinger field theory. The ETCCR are

$$[\psi(\mathbf{x}, t), \psi^\dagger(\mathbf{y}, t)] = \delta^{(3)}(\mathbf{x} - \mathbf{y}), \qquad [\psi(\mathbf{x}, t), \psi(\mathbf{y}, t)] = 0, \tag{1}$$

and the formula for H in terms of the quantum fields was given in class.

In this problem you are to stay within the field theory formalism. The ETCCR and H are specified at $t = 0$, and the time-dependence of the fields is given by the Heisenberg equation of motion. The aim is to show that the ETCCR and the formula for H continue to hold at all other time.

3. In the theory of a Schrödinger field ψ, define an operator $A(t)$ by

$$A = \int \mathrm{d}^3\mathbf{x}\, x_1\, \psi^\dagger(\mathbf{x}, t)\, |0\rangle\, \langle 0|\, \psi(\mathbf{x}, t). \tag{2}$$

Here position vectors are written in terms of Cartesian coordinates as $\mathbf{x} = (x_1, x_2, x_3)$. Don't forget to notice the factor of x_1 in this definition.

(a) Show that A gives zero on the vacuum and on states of two or more particles: it can only be nonzero on states of one particle.

(b) Now apply this operator to a one-particle state $|f\rangle$ with wave function $f(\mathbf{x})$ at time t:

$$|f\rangle \equiv \int \mathrm{d}^3\mathbf{x}\, \psi^\dagger(\mathbf{x}, t)\, |0\rangle\, f(\mathbf{x}). \tag{3}$$

Obtain the wave function (at time t) of the state $A\,|f\rangle$.

(c) What well-known operator in Schrödinger wave mechanics coincides with A?