Physics 563 (Quantum Field Theory 1) Fall 2009

Problem set 3

Due Sep. 23, 2009

1. The standard units ('natural units') used by elementary particle theorists have $\hbar = c = \epsilon_0 = 1$, and have their unit of energy equal to $1\,\mathrm{GeV}$. In this system of units, obtain the dimensions of length, area, time, electric potential and electric charge? (I.e., what dimension are they in powers of GeV?)

For each of these quantities, when it has a unit value in natural units, obtain its numerical value in SI units. (E.g., when a length L has a value $L = 1\,\mathrm{GeV}^{-1}$ — or more strictly $L = 1\,\hbar c\,\mathrm{GeV}^{-1}$ — what is it value in meters?)

What is the conversion factor to go from a cross section calculated in units of GeV^{-2} to a value in mb? (1 mb is 1 millibarn, and a barn is defined to be $10^{-24}\,\mathrm{cm}^2$.)

What is the charge of the electron in natural units?

NOT GRADED 2. Let A and B be operators whose commutator $[A, B] = c$ is a pure number. Let $f(B)$ be an operator that is written as a polynomial in terms of B. Prove that

$$[A, f(B)] = c\frac{\partial f(B)}{\partial B}. \tag{1}$$

3. *Invariance of Hamiltonian formalism under change of variable.* Consider a Lagrangian of the form

$$L = \tfrac{1}{2}\sum_{i,j} \dot{q}^i K_{ij} \dot{q}^j - \tfrac{1}{2}\sum_{i,j} q^i V_{ij} q^j. \tag{2}$$

Here there are N generalized coordinates q^i, K is a real symmetric positive-definite[1] matrix, and V are real symmetric positive-semidefinite[2] matrix. Let the corresponding momenta be p_i, and find them in terms of the $\dot{q}^i$. Work out the Hamiltonian, and write down the ETCCR[3] for the case that the system is treated as quantum mechanical.

Now change variables to

$$x^i = \sum_j S^i{}_j q^j, \tag{3}$$

where S is a non-singular $N \times N$ matrix.

[1] An $N \times N$ real square matrix M is defined to be *positive-definite* when $x^T M x > 0$ for every nonzero N-dimensional real column vector x. Sometimes the condition is restricted to symmetric matrices. It is readily generalized to complex square matrices, especially Hermitian matrices.

[2] An $N \times N$ real square matrix M is defined to be *positive-semidefinite* when $x^T M x \geq 0$ for every nonzero N-dimensional real column vector x. Normally the condition is restricted to symmetric matrices. It is readily generalized to complex square matrices, normally Hermitian.

[3] Equal-time canonical commutation relations.

(a) What is the Lagrangian in terms of these new coordinates?

(b) From the new Lagrangian, find the momenta π_i corresponding to the x coordinates, and express them in terms of the p_i.

(c) What is the Hamiltonian for the new coordinates and momenta?

(d) Hence find the ET commutation relations for the new coordinates and momenta (x and π) from the ETCCR for the old coordinates and momenta (q and p). These should have the form of the standard ETCCR, i.e., the quantum mechanics formalism is consistent between the two coordinate systems

(e) Are there any other consistency requirements between the two coordinate systems that need to be checked?

4. *Fundamental results for systems with a quadratic lagrangian.* Consider again the theory defined by the Lagrangian in (2), with the potential-energy matrix V now required to be positive-definite instead of merely positive-semidefinite.

(a) Show that a choice of change of variable can be made, of the form of Eq. (3), so that the Lagrangian is a sum of N independent SHOs:

$$L = \sum_\alpha \left[\tfrac{1}{2}(\dot{Q}^\alpha)^2 - \tfrac{1}{2}\omega_\alpha^2 (Q^\alpha)^2 \right]. \tag{4}$$

(Hint apply successive changes of variable to obtain this formula, and note that the matrix S need *not* be orthogonal.)

(b) What is the most general solution (classically) for Q and hence for q?

(c) Hence show that there are exactly N independent normal modes, and find the orthogonality condition.[4]

(d) Write out the general formula for $q^i(t)$ as a sum of normal modes times coefficients a_α and $a_\alpha^\dagger$. Now using the same formula in quantum mechanics, show that normalizations can be adjusted so that these operators obey the standard commutation relations

$$[a_\alpha, a_\beta] = [a_\alpha^\dagger, a_\beta^\dagger] = 0, \quad [a_\alpha, a_\beta^\dagger] = \delta_{\alpha\beta}. \tag{5}$$

(e) Show that the Hamiltonian can be written as a sum of independent harmonic oscillators

$$H = \sum_\alpha \hbar\omega_\alpha (a_\alpha^\dagger a_\alpha + \tfrac{1}{2}). \tag{6}$$

[4] An orthogonality condition means the following. Let a normal mode of label α be specified by a set of coefficients ${X_\alpha}^j$ for which $q^j(t) = {X_\alpha}^j e^{-i\omega_\alpha t}$ gives a solution of the equations of motion. Then an orthogonality condition is of the form $\sum_{ij} {X_\alpha}^i K_{ij} {X_\beta}^j = \delta_{\alpha\beta} f_\alpha$, with some matrix K_{ij} that you should give a formula for.

Note: Try and be as creative as possible to simplify the argument!

5. Let $\phi_a(t)$ be a massless ($m = 0$) Klein-Gordon field averaged with a function proportional e^{-r^2/a^2}, where r is the distance is from the origin of spatial coordinates. That is

$$\phi_a(t) = \frac{\int \mathrm{d}^3\mathbf{r}\, \phi(\mathbf{r}, t) e^{-r^2/a^2}}{\int \mathrm{d}^3\mathbf{r}\, e^{-r^2/a^2}} = \frac{\int \mathrm{d}^3\mathbf{r}\, \phi(\mathbf{r}, t) e^{-r^2/a^2}}{\pi^{3/2} a^3}. \tag{7}$$

You can think of $\phi_a(t)$ as a smeared version of the field, with a resolution of a.

Compute the vacuum expectation value of ϕ_a^2, i.e.,

$$\langle 0|\phi_a^2|0\rangle. \tag{8}$$

(Use natural units!) The square root of this expectation value is an estimate of the fluctuations in the field ϕ when probed with a detector with resolution a. Convert this quantity to *volts*. (I think you should multiply by $\sqrt{\hbar c/\epsilon_0}$, but you should check!) This estimate should also be good for the electromagnetic field, to within a modest factor. Compute numerical values for a few distance scales of physical interest.

In what situations might these 'zero point fluctuations' be of significant? Try and indicate why.

6. Let

$$f(\mathbf{k}) \equiv Ce^{-(\mathbf{k}-\mathbf{l}_1)^2/\Delta^2} \tag{9}$$

$$g(\mathbf{k}) \equiv De^{-(\mathbf{k}-\mathbf{l}_2)^2/\Delta^2} \tag{10}$$

be momentum space wave functions for a single particle. Neglect their overlap, i.e., assume $\Delta \ll |\mathbf{l}_1 - \mathbf{l}_2|$. You should adjust C and D to suitable values so that the corresponding one-particle states are normalized, i.e., $\langle f|f\rangle = \langle g|g\rangle = 1$. Don't bother to evaluate any but the simplest integrals.

For each of the following cases, find a formula involving an integral over *free Klein-Gordon* creation operator(s) applied to the vacuum state:

(a) A normalized one-particle state corresponding to f.

(b) A normalized one-particle state which has 10% probability of being in the f state and 90% for the g state.

(c) A normalized two-particle state where one particle is in the f state and one in the g state.

(d) A normalized two-particle state where both particles are in the same f state. (N.B. Check your normalizations carefully. This part has a trap!)

(e) Are the solutions to these requirements unique?