Problem set 4

Due Oct. 9, 2009

NOT GRADED 1. *Normal modes for lattice Klein-Gordon theory* Let 3-space be a cubic lattice with points

$$\mathbf{x}_{\mathbf{n}} = a(n_1, n_2, n_3), \tag{1}$$

where a is the lattice spacing, and the n_js are integers. There are N points in each coordinate direction, for a total volume of space of $V = a^3 N^3$. Apply periodic boundary conditions.

The Lagrangian is

$$L = \sum_{\mathbf{n}} a^3 \left[\frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \sum_{j=1,2,3} (D_j \phi)^2 - \frac{m^2}{2} \phi^2 \right]. \tag{2}$$

Here the field $\phi(t, \mathbf{x})$ is defined on the lattice points, and $D_j \phi$ denotes a discrete approximation to the spatial derivative, e.g.,

$$D_1 \phi = \frac{\phi(t, a(n_1 + 1, n_2, n_3)) \ - \ \phi(t, a(n_1, n_2, n_3))}{a}. \tag{3}$$

Find the equation of motion of the field, which should be a discretized version of the Klein-Gordon equation.

Find the most general solution, which should be of the form

$$\phi = \sum_{\alpha} \left(\hat{a}_\alpha e^{-i\omega_\alpha t + i\mathbf{k}_\alpha \cdot \mathbf{x}} + \hat{a}_\alpha^\dagger e^{i\omega_\alpha t - i\mathbf{k}_\alpha \cdot \mathbf{x}} \right). \tag{4}$$

You should find that the momenta are of the form

$$\mathbf{k} = \frac{2\pi}{aN} \ (j_1, j_2, j_3), \tag{5}$$

where the j_ks are integers, and that the normal mode angular frequencies are

$$\omega = \sqrt{m^2 + \sum_{j=1,2,3} \frac{4}{a^2} \sin^2 \frac{k_j a}{2}}. \tag{6}$$

NOT GRADED 2. Define a theory by

$$L = \frac{1}{2} \left(q_1 \dot{q}_2 - q_2 \dot{q}_1 \right) - f(q_1, q_2), \tag{7}$$

where $q_1(t)$ and $q_2(t)$ are real/hermitian coordinates. For this problem, use

$$f(q_1, q_2) = \frac{\omega}{2} \left({q_1}^2 + {q_2}^2 \right). \tag{8}$$

- Find the Euler-Lagrange equations of motion.
- Apply the Faddeev-Jackiw quantization rules to find the ETCR and the Hamiltonian.
- Show that the standard Heisenberg rules give the correct equations of motion.
- Find the solution given the formula for $f(q_1, q_2)$.
- Show that the result is the usual simple harmonic oscillator in an unusual formulation.

3. Now define a theory by

$$L = \frac{i}{2}\left(\psi^\dagger \dot{\psi} - \dot{\psi}^\dagger \psi\right) - \omega \psi^\dagger \psi, \tag{9}$$

where $\psi(t)$ is a *complex* coordinate. Apply the Faddeev-Jackiw rules for quantization in their form for complex coordinates, as in the Schrödinger field theory. Show that the results are equivalent to those in the previous problem with a suitable redefinition of variables.