Physics 563 (Quantum Field Theory 1) Fall 2009

Problem set 5

Due Oct. 21, 2009

1.

(a) Find an ordinary wave function that represents a hydrogen atom in its ground state with definite total momentum.

(b) Now consider a Schrödinger-type field theory, with an electron and a proton field (and a Coulomb potential between them. Find a formula for the state in the previous part. Your answer should be in the form of an integral over $\psi_e^\dagger(t, \mathbf{x})\, \psi_p^\dagger(t, \mathbf{y})\, |0\rangle$, where ψ_e and ψ_p are Schrödinger fields for the electron and proton.

(c) Should the integral be symmetrized, or not? Why?

(d) What if the atom is in the $n = 2$, $l = 1$, $m = 1$ state? (Here n, l, m are the usual quantum state labels for states of a hydrogen atom.)

2. *Noether's theorem for particle mechanics: general case.* In class, I derived Noether's theorem for a theory defined by a Lagrangian density $\mathcal{L}$. Now consider an ordinary Lagrangian $L(q, \dot{q})$. For example, for a nonrelativistic particle in a spherically symmetric potential we have

$$L = \frac{1}{2} m \dot{\mathbf{x}}^2 - V(|\mathbf{x}|). \tag{1}$$

Consider a general Lagrangian $L(q, \dot{q})$, where q is a multidimensional generalized coordinate variable. Suppose that it is invariant up to a total time derivative when q under a certain group of transformations with parameters ω_α. Let the linearized form of the transformations be $q^j \mapsto q^j + \omega_\alpha (D^\alpha q)^j + O(\omega^2)$, and let the resulting transformation of the Lagrangian be $L \mapsto L + \omega_\alpha \frac{\mathrm{d}}{\mathrm{d}t} f^\alpha + O(\omega^2)$. Here the f^αs are some functions of the coordinates and their derivatives. [Note that $(D^\alpha q)^j$ represents the derivative of the transformed coordinate with respect to ω_α.]

State and prove the version of Noether's theorem that applies to such a cases. You will not have conserved currents, but just conserved charges.

3. *Noether's theorem for particle mechanics: example.* The above Lagrangian (1) is invariant under rotations about the origin. We parameterize a rotation by a 3-vector $\boldsymbol{\omega}$ whose length is the angle θ of the rotation, and whose direction is the axis of the rotation. Then infinitesimal rotations are given by

$$\omega_i\, D^i \mathbf{x} = \boldsymbol{\omega} \times \mathbf{x}. \tag{2}$$

Show that that the Lagrangian is in fact invariant. Then find the conserved charges, which should just be the usual angular momentum operators.

NOT GRADED 4. *This problem is just for you to make sure you can reproduce results obtained in class.*
"Isospin" symmetry A particular quantum field theory has two complex scalar fields ϕ_1 and ϕ_2, which can be conveniently combined into a column vector:

$$\phi(x) = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}. \tag{3}$$

The Lagrangian is

$$\mathcal{L} = -\partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2. \tag{4}$$

Here the two-component conjugate field $\phi^\dagger$ is $(\phi_1^\dagger, \phi_2^\dagger)$, i.e., a row vector with conjugated fields. Thus $\phi^\dagger \phi$ means $\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2$.

(a) Verify that $\mathcal{L}$ is invariant under the group of SU(2) transformations (called "isospin") which transform ϕ_j to

$$\left(e^{-i\omega_a \sigma_a/2} \right)_{jj'} \phi_{j'}, \tag{5}$$

where ω_1, ω_2 and ω_3 are the parameters of the group and σ_1, etc are the Pauli matrices. (Summation convention understood, of course.) N.B. Recall (probably from classes on spin-$\frac{1}{2}$ particles in quantum mechanics) that the 2×2 matrix $e^{-i\omega_a \sigma_a/2}$ is unitary, and of determinant unity. But note that the label on the field is not intended to refer to spin.

(b) What are the "infinitesimal" transformations $D_a \phi_j$? That is, what are the derivatives of the transformed fields with respect to the parameters at zero ω?

(c) Obtain the Noether currents $j_{a\mu}$ and charges Q_a for the transformations.

5. *Properties of the charges correspond to generators of the group.* In the same field theory as in the previous problem:

(a) Verify from the canonical commutation relations of the fields that $[Q_a, \phi_j]$ gives the infinitesimal transformation, with an appropriate factor of $\pm i$.

(b) Show that the charges obey the SU(2) algebra $[Q_a, Q_b] = i \sum_c \epsilon_{abc} Q_c$. (Here ϵ_{abc} has its usual meaning.)

N.B. The commutation relations you derived are the same as those obeyed by the matrices $\sigma_a/2$ used in the original specification (5) of the transformations of the field. Thus you have proved/verified that the operators Q_a on state space give a representation of the Lie algebra of the matrices $\sigma_a/2$.

Do your best to find an economical way of presenting you solution. Otherwise, there will be a lot of algebra.