Path/functional integral

- For both quantum mechanics of particles and QFT
- See
  - Srednicki Ch. 6–10.
  - R. MacKenzie, "Path Integral Methods and Applications", `http://arxiv.org/abs/quant-ph/0004090`
  - R. Feynman, "Space-Time Approach to Non-Relativistic Quantum Mechanics", Rev. Mod. Phys. **20**, 367–387 (1948)
- Start with QM of one particle in 1-dim

$$\langle q''; t'' \,|\, q'; t' \rangle = \left\langle q'' \,\middle|\, e^{-iH(t''-t')} \,\middle|\, q' \right\rangle_{\text{Schr. pic.}}$$

$$= \text{norm.} \times \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}q \; e^{iS[q]}$$

with $\mathcal{D}q \;=\; \prod_t \mathrm{d}q(t)$

- Proof (rigor?):
  - Split time range into small intervals
  - Integral over complete set of states is integral over configurations:

$$\langle B \,|\, A \rangle = \int \mathrm{d}q \, \langle B \,|\, q \rangle \, \langle q \,|\, A \rangle$$

  - Use

$$\left\langle q'' \,\middle|\, e^{-i\delta t \hat{p}^2/(2m)} \,\middle|\, q' \right\rangle \propto \exp\left[+i \frac{m(q''-q')^2}{2\delta t} \right]$$

  - Then insert $e^{-iV(\hat{q})\delta t}$ factors and approximate for small δt.
- Tricky problem: Convergence of oscillating integrals.
- Solve by "Wick rotation" to imaginary time: $t = -i\tau$

Related results

- Matrix elements with operators

$$\left\langle q'';t'' \right| \; T\hat{q}(t_1)\ldots\hat{q}(t_N) \; \left| q';t' \right\rangle = \text{norm.} \times \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}q \; q(t_1)\ldots q(t_N) \; e^{iS[q]}$$

$$\left\langle q'';t'' \right| \; TF[\hat{q}] \; \left| q';t' \right\rangle = \text{norm.} \times \int_{q(t')=q'}^{q(t'')=q''} \mathcal{D}q \; F[q] \; e^{iS[q]}$$

- Euclidean (imaginary time) = partition function. Use $q(t) = q(-i\tau)$

$$\mathrm{tr}\, e^{-H\beta} = \text{norm.} \times \int_{\text{Eucl., periodic}} \mathcal{D}q \; e^{-S_E[q]}$$

$$S_E[q] = \int_0^\beta \mathrm{d}\tau \left(\frac{m}{2} \left(\frac{\mathrm{d}q}{\mathrm{d}\tau} \right)^2 + V(q) \right)$$

- Green function (after proper limit)

$$\langle 0 \,|\, T\hat{q}(t_1)\ldots\hat{q}(t_N) \,|\, 0 \rangle = \text{norm.} \times \int \mathcal{D}q \; q(t_1)\ldots q(t_N) \; e^{iS[q]}$$