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The role of analogy in unraveling the fractional quantum Hall effect mystery

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Abstract

This introductory article, intended for non-experts, explains how the analogy to the integral quantum Hall effect provided the key to understanding the fractional quantum Hall effect, at the same time leading to a new class of topological fermions called composite fermions. It is shown how the enormous ambiguity of the original electron problem is eliminated as a result of the formation of composite fermions, and a unique solution is obtained that is extremely accurate and consistent with dramatic experimental observations.

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Professor von Klitzing's seminal discovery [1] of the integral quantum Hall effect (IQHE) set in motion a chain of events that has resulted in the uncovering of some of the most elegant emergent structures found in a many-particle system. This article will emphasize how the IQHE played another crucial role in the development of the field, by providing the analogy that helped uncover the highly non-perturbative physics of the fractional quantum Hall effect (FQHE) [2]. I will take this opportunity to write a pedagogical article, intended for non-specialists, to explain in a simple fashion what this analogy is and how it works. Only the most basic physics is given here; for further information, the reader may find Refs. [3,4] to be useful starting points.

1. Landau levels

The system of interest contains electrons confined in two dimensions and exposed to a strong magnetic field, the physics of which is governed by the Hamiltonian

$$H=\sum_j \frac{1}{2m_e}\left[\frac{\hbar}{i}\nabla_j+\frac{e}{c}\boldsymbol{A}(\boldsymbol{r}_j)\right]^2 +\sum_{j<k}\frac{e^2}{|\boldsymbol{r}_j-\boldsymbol{r}_k|}. \qquad (1)$$

As we will see, this very simple looking problem contains much physics.

For $H=(\boldsymbol{p}+e\boldsymbol{A}/c)^2/2m_e$, a single electron in a magnetic field, the solution is straightforward. The kinetic energy is quantized at $E=(n+1/2)\hbar eB/m_ec$, where the discrete energy levels labeled by $n=0,1,\ldots$ are called Landau levels. The degeneracy of each Landau

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level is given by

$$M = \frac{BA}{\phi_0}, \qquad (2)$$

where B is the external field, A is the area of the sample, and $\phi_0 = hc/e$ is called the "flux quantum."

2. IQHE

In 1980 von Klitzing discovered, completely unexpectedly, that the Hall resistance of a two-dimensional system has plateaus where its value is quantized at

$$R_{\mathrm{H}} = \frac{h}{ne^2}, \qquad (3)$$

where n is an integer. This phenomenon is known as the integral quantum Hall effect. The value of the quantized Hall resistance is independent of the sample type, geometry, sample-specific parameters (for example, the dielectric constant or the electron band mass), or disorder. The relative accuracy of the quantization of R_{H} has been established [5] to a few parts in 10^{10}, which is equivalent to knowing the earth's radius with an accuracy of a couple of millimeters. It is believed that the quantization is exact, and the ratio $R_{\mathrm{K}} = h/e^2$, called the von Klitzing constant, has been adopted as the fundamental unit of resistance. The Hall experiment on a dirty solid-state system provides one of the most accurate values for the fine structure constant $\alpha = e^2/\hbar c$.

The origin of the IQHE was soon clarified [6], because, as it happens, the effect also occurs in a theoretical model of *non*-interacting electrons. To see the physics, let us imagine a hotel, "The Landau Tower," with an infinite number of floors, and with M rooms on each floor. The different floors represent Landau levels, and the number of rooms on each floor, M, is the degeneracy of a Landau level. (There is precisely one room per quantum of flux penetrating the sample.) It should be kept in mind that the number of rooms on each floor is not fixed but increases with increasing magnetic field.

The occupants of the rooms are, of course, the electrons. As decreed by Pauli, multiple occupancies are strictly forbidden in this hotel. (We will assume that all the electrons have the same spin orientation, so we do not have to worry about the spin degree of freedom.) Electrons want to check into the lowest available floors, to save themselves the work required to climb up, so the floors get filled from the ground up. The number of filled floors,

$$\nu = \frac{N}{M} = \frac{N\phi_0}{AB}, \qquad (4)$$

where N is the number of electrons, is called the filling factor.

In general, when the highest occupied floor is only partially full, there is a lot of freedom about which rooms electrons occupy. But when the number of electrons is $N = nM$, or the filling factor is an integer ($\nu = n$), a unique arrangement is obtained in which the lowest n floors are fully occupied.

What we have described can be viewed as a giant atom containing N ($\sim 10^9$) electrons, with quantized kinetic energy shells (Landau level floors) of degeneracy M. The atoms with an integral number of filled shells are especially simple, akin to the noble gas atoms: the ground state wave function is completely known (a Slater determinant), and there is a gap to excitations. This uniqueness of the ground state at integral fillings lies at the heart of the IQHE.

As explained by Laughlin [6], disorder-induced Anderson localization also plays a crucial role in the establishment of the Hall plateaus. To see this, imagine changing the filling away from an integer by adding some electrons or holes. In a perfect system, the additional particles would also be free to carry current, but in the actual, disordered sample, they are immobilized by impurities (which create localized states in the energy gap), and do not contribute to transport. The transport properties therefore remain unaffected as the filling factor is varied slightly away from an integer, and the system continues to behave as though it had filled shells. This, crudely, is how the presence of a gap at filling $\nu = n$ in a pure system becomes manifest, in the presence of disorder, through a quantized Hall plateau at $R_{\mathrm{H}} = h/ne^2$.

3. FQHE

The next revolution occurred when Tsui, Stormer, and Gossard [2] discovered the fractional quantum Hall effect. The 1982 observation of a plateau at $R_{\mathrm{H}} = h/(1/3)e^2$ gave us the first glimpse into what would eventually be recognized as one of the most amazing collective states of matter in nature. In the subsequent years, as experiments improved, there was an explosion in the number of fractions, as seen in

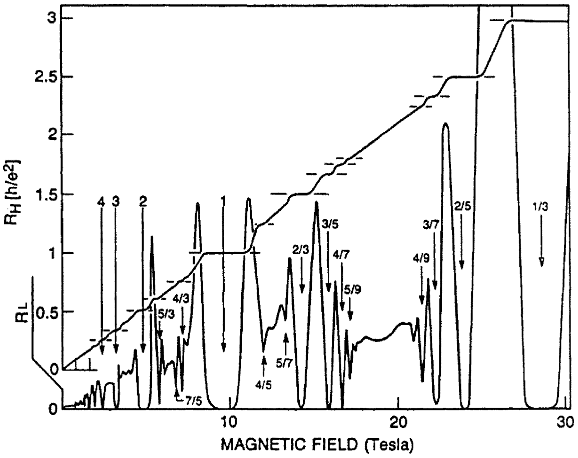

Fig. 1. The stunning QHE skyline. Associated with the quantized plateaus on the Hall resistance $R_{\rm H}$ are minima in the longitudinal resistance $R_{\rm L}$, where $R_{\rm L}$ vanishes exponentially in the limit of vanishing temperature. The integers/fractions corresponding to the minima or the plateaus are shown in the figure. *Source*: H.L. Stormer and D.C. Tsui, in Ref. [3].

Fig. 1. Each fraction f reveals itself through a quantized Hall plateau at

$$R_{\rm H} = \frac{h}{fe^2} \tag{5}$$

centered at $\nu = f$. Many more fractions have been observed since Fig. 1, and today there exists evidence for more than 50 fractions with $f < 1$. The mind-boggling abundance of structure for a single resistance measurement represents an extraordinarily exciting new effect.

As the fractions proliferated, at first the situation grew more confusing, as usual, but then certain patterns emerged, which eventually provided the key to understanding the effect. Many fractions were conspicuous by their absence. For example, the simplest fraction $f = \frac{1}{2}$ was not observed. In fact, no fractions with even denominators were observed. Another striking aspect was that the fractions were not isolated but belonged to certain sequences. For example, in Fig. 1, the fractions $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{5}{11}, \cdots$ are seen to follow the sequence $f = n/(2n+1)$.

4. World's most strongly correlated quantum fluid

The questions that theory was challenged to answer could not be more sharply posed. What physical principle governs this quantum fluid? What describes the order that makes the ground state unique and special, as the experiments are telling us, at certain fractional fillings? What are these ground states? Why are some fractions chosen and not others? Why do they appear in certain sequences? Obviously, these questions are not independent, but must be answered all at once.

The FQHE occurs in very strong magnetic fields. Because the Landau level spacing is the dominant energy in this limit, electrons first minimize their kinetic energy by all occupying the lowest Landau floor. The kinetic energy thus, becomes a constant and drops out of the problem. The Coulomb interaction is then the only remaining term. With appropriate units for energy and length, the Hamiltonian simplifies to

$$H = \sum_{j<k} \frac{1}{|\boldsymbol{r}_j - \boldsymbol{r}_k|}, \tag{6}$$

which is to be solved in the lowest Landau level Hilbert space, that is, when only the lowest floor of The Landau Tower is occupied.

In many-body problems, the starting point for a theoretical investigation of a phenomenon is often obtained by switching off the interaction altogether. But the interaction is not small compared to anything else in our problem, because there is nothing else. The interaction is the only energy in our problem, and switching it off is not a good idea. Eq. (6) describes a very strong correlated system, because it contains no small parameter. (There is no parameter, period.) Alternatively, one may note that the usual quantity characterizing the strength of the correlations, namely, the ratio of the interaction energy to the kinetic energy, is $r_{\rm s} = \infty$.

Let us ask what happens if, just for a moment, we set the interaction to zero. In that case, there are a large number of equivalent states, because there are many more rooms than there are electrons. To get a glimpse into the enormous complexity of the problem, let us do a simple calculation. A typical 1 mm × 1 mm sample contains 10^9 electrons. For 2.5×10^9 rooms ($\nu = 0.4$), there are $10^{7\times 10^8}$ distinct configurations. Even for a toy system containing only 100 electrons in 250 rooms, there are 10^{72} distinct configurations (which is roughly equal to the number of atoms in the universe).

Clearly, in the absence of interaction, no single state is picked out at any fractional filling factor. When the interaction is switched back on, the configurations are no longer equivalent; the degeneracy is lifted and one of the states becomes the ground state. (Because electrons are quantum mechanical objects, nature is not restricted to a single configuration with some rooms occupied and the rest vacant, but will choose for the ground state a linear superposition of a great many such configurations.) The difficulty is that, on purely theoretical grounds, with no small parameter to guide our thinking, we have no clue where to start looking for the ground state. (Crystalline arrangements of electrons can be quickly ruled out, as experiments firmly point to a liquid.) Standard perturbative methods are of no use and the problem appears hopelessly intractable.

5. The analogy

Confronted with an impasse, we search for an analogy. What is the key analogy here? Let us look again at Fig. 1 and, to avoid distraction with too many details, do the mental exercise of erasing all the numbers. It then jumps out that it is hard to tell the integer plateaus from the fractional ones. Our search for an analogy that would provide an insight into the FQHE thus brings us all the way back to square one: the IQHE. Can the well understood IQHE teach us something about the mysterious fractional quantum Hall effect? As we will see, the answer is in the affirmative, and the relation between the two is profound and far-reaching. In hindsight, that is not surprising at all. The history of physics is replete with examples where analogies crop up in unexpected places, but here, the two phenomena, the integral and the fractional Hall effects, look so similar that it would really be inconceivable for them not to be related.

In view of the remarkable parallel between the FQHE and the IQHE, it is tempting to postulate that the FQHE is really an IQHE in disguise. That, however, raises the question: Of *what* is it the IQHE? What objects in the FQHE are analogous to the electrons of the IQHE? One's natural expectation is that these should be some kind of fermions, because it was the fermionic nature of the electrons that made integer fillings special. The pursuit of the analogy between the FQHE and the IQHE thus points inescapably to the existence of a new kind of fermionic particles in the FQHE.

What are these fermions? To recall an example from a different context, the similarity between superconductors and the ^{4}He superfluid suggests that the superconductors contain certain bosonic objects, which are the familiar Cooper pairs containing two electrons each. For our problem, one might envision new fermions arising through formation of bound states containing an odd number of electrons. However, the occurrence of such bound states appears unlikely because the interaction between electrons is repulsive; a little further thought shows this scenario to be grossly inconsistent with the known phenomenology, thus ruling it out as a viable possibility.

6. Composite fermion: an intuitive view

It turns out that the new fermions, called "composite fermions", [7] are certain incredibly complicated entities, with each composite fermion made up of *all* electrons. The emergence of such a complex object is a testament to the profoundly collective nature of this quantum fluid. Fortunately, there exists a simple, intuitive way of viewing composite fermions which is very useful and gives the basic physics and some of the relevant equations in a single particle language. That is what we first present. (We caution the reader that this simple picture is not to be taken too literally. The more accurate description will follow in the next section.)

Fig. 2 shows the principal idea. Begin with electrons in the presence of an external magnetic field B, at a filling ν. Now picture that each electron "absorbs" an even number $(2p)$ of flux quanta of the external magnetic field. Why it does so and what that really means will become clear only when we come to the microscopic theory. The bound state of an electron and $2p$ flux quanta is known to obey fermionic

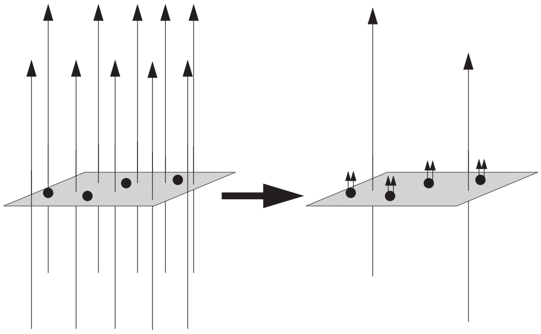

Fig. 2. A schematic depiction of how electrons capture flux quanta to transform into composite fermions, which experience a smaller, residual magnetic field.

statistics, and serves as an approximate representation of a composite fermion. The even integer $2p$ can be called the "flux quantum number" (or, for reasons explained later, the "vortex quantum number") of the composite fermion. Further, assume that the interaction between composite fermions is very weak and, in the first approximation, can be neglected altogether.

The most fundamental property of composite fermions is that they experience a reduced residual magnetic field (called B^*), the flux for which is given by

$$AB^* = AB - 2pN\phi_0, \tag{7}$$

where the last term is equal to the magnetic flux that has been integrated into the composite fermions. (Note that B^* can have a negative sign, which occurs when electrons swallow more flux than there was to begin with.) Now, in complete analogy to electrons, the composite fermions have their own shells, but with the degeneracy of each shell given by

$$M^* = \frac{A|B^*|}{\phi_0} = |M - 2pN|, \tag{8}$$

which implies that the filling factor of composite fermions is

$$v^* = \frac{N}{M^*}. \tag{9}$$

The value of the integer p is chosen to minimize M^* (which will give $M^* \leqslant N$, or $v^* \geqslant 1$) because the fewer the rooms, the smaller is the ambiguity. This brings us to a new problem, that of N composite fermions in a different tower, again with infinite floors, but with M^* rooms on each floor.

What happens next is magic. Consider

$$v^* = n. \tag{10}$$

Here, the state of composite fermions has n floors fully occupied. This is the IQHE of composite fermions. A little algebra shows that the filling factors $v^* = n$ of composite fermions correspond to the electron filling factors

$$v = \frac{n}{2pn \pm 1}, \tag{11}$$

where the $-$ sign corresponds to the situation when B^* points opposite to B. At these filling factors, when we reformulate the problem in terms of composite fermions, the enormous ambiguity of the electron problem suddenly disappears to produce a single, well-defined state. (Fig. 3.) As in IQHE, the gaps at these filling factors lead, in the presence of disorder, to quantized Hall plateaus at $R_H = h/fe^2$, with

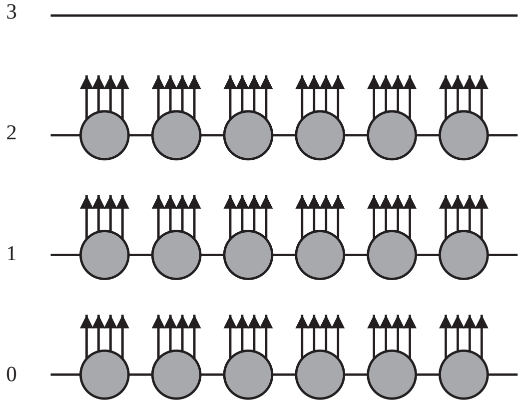

Fig. 3. The schematic energy level diagram for the ground state at $v = 3/13$, labeled ${}^4CF_3$, which has three filled shells of composite fermions carrying four flux quanta (vortices) each.

$$f = \frac{n}{2pn \pm 1}. \tag{12}$$

These are precisely the prominently observed fractions (Table 1). They have odd denominators, and they occur in sequences because they are obtained from the sequence of integers. The ordering of fractions is consistent with experiment. For composite fermions of a given flux ($2p$) the fraction with fewer filled shells is more robust, and for a given number of filled shells $|n|$ the fraction for simpler particles (smaller $2p$) is more robust. In other words, Table 1 gets filled without vacancies.

Let us come back to our simple-minded description of electrons in a hotel, keeping in mind the rule that when electrons convert into composite fermions by capturing $2p$ flux quanta, the number of rooms on each floor is reduced by $2p$ per particle. Consider a system of four electrons with nine rooms on each floor, representing $v=4/9$. With $2p=2$, the four composite fermions which replace the four electrons, see a hotel with only one room per floor, and hence occupy the first four floors to produce the unique ground state of the system. To take another example, consider three electrons with 11 rooms on each floor, which corresponds to $v = 3/11$. Here, if we choose composite fermions with $2p = 2$ attached flux quanta, the three composite fermions see five rooms on each floor, which reduces the ambiguity but does not eliminate it. On the other hand, with $2p = 4$, four rooms per

Table 1
The table of prominently observed fractions in the lowest electronic Landau level, which are associated with giant, filled-shell atoms of composite fermions

IQHE of composite fermions

$\lvert n\rvert$	1	2	3	4	5	6	7	8	9	10	11
f	1/3	2/5	3/7	4/9	5/11	6/13	7/15	8/17	9/19	10/21	
state	$^{2}\mathrm{CF}_{1}$	$^{2}\mathrm{CF}_{2}$	$^{2}\mathrm{CF}_{3}$	$^{2}\mathrm{CF}_{4}$	$^{2}\mathrm{CF}_{5}$	$^{2}\mathrm{CF}_{6}$	$^{2}\mathrm{CF}_{7}$	$^{2}\mathrm{CF}_{8}$	$^{2}\mathrm{CF}_{9}$	$^{2}\mathrm{CF}_{10}$	
f	1/5	2/9	3/13	4/17	5/21	6/25					
state	$^{4}\mathrm{CF}_{1}$	$^{4}\mathrm{CF}_{2}$	$^{4}\mathrm{CF}_{3}$	$^{4}\mathrm{CF}_{4}$	$^{4}\mathrm{CF}_{5}$	$^{4}\mathrm{CF}_{6}$					
f	1/7	2/13	3/19								
state	$^{6}\mathrm{CF}_{1}$	$^{6}\mathrm{CF}_{2}$	$^{6}\mathrm{CF}_{3}$								
f	1/9	2/17									
state	$^{8}\mathrm{CF}_{1}$	$^{8}\mathrm{CF}_{2}$									
f	1	2/3	3/5	4/7	5/9	6/11	7/13	8/15	9/17	10/19	
state	$^{2}\mathrm{CF}_{-1}$	$^{2}\mathrm{CF}_{-2}$	$^{2}\mathrm{CF}_{-3}$	$^{2}\mathrm{CF}_{-4}$	$^{2}\mathrm{CF}_{-5}$	$^{2}\mathrm{CF}_{-6}$	$^{2}\mathrm{CF}_{-7}$	$^{2}\mathrm{CF}_{-8}$	$^{2}\mathrm{CF}_{-9}$	$^{2}\mathrm{CF}_{-10}$	
f	1/3	2/7	3/11	4/15	5/19	6/23					
state	$^{4}\mathrm{CF}_{-1}$	$^{4}\mathrm{CF}_{-2}$	$^{4}\mathrm{CF}_{-3}$	$^{4}\mathrm{CF}_{-4}$	$^{4}\mathrm{CF}_{-5}$	$^{4}\mathrm{CF}_{-6}$					
f	1/5	2/11	3/17								
state	$^{6}\mathrm{CF}_{-1}$	$^{6}\mathrm{CF}_{-2}$	$^{6}\mathrm{CF}_{-3}$								
f	1/7	2/15									
state	$^{8}\mathrm{CF}_{-1}$	$^{8}\mathrm{CF}_{-2}$									

Columns and the rows are arranged according to the number of filled shells ($|n|$) and the flux (vortex) quantum number ($2p$) of the relevant composite fermions. The filled shell states associated with the fractions $f = n/(2pn+1)$ are denoted by $^{2p}\mathrm{CF}_n$; negative values of n, which produce fractions $f = |n|/(2p|n|-1)$, refer to the situation in which B^* is negative. When a fraction appears in more than one place, the description in terms of the simpler particles is to be taken. Many more fractions follow from those given in this table, which are not shown for simplicity. For example, particle–hole symmetry in the lowest Landau level implies fractions $f = 1 - n/(2pn \pm 1)$. These fractions can be obtained by defining the original problem in terms of holes, rather than electrons, in the lowest Landau level and then making composite fermions out of them. In addition to the fractions that are manifestations of the *integral* QHE of composite fermions, a few secondary fractions originating from the *fractional* QHE of composite fermions have also been detected (not shown). Fractions observed in higher Landau levels are also not listed. For fractions beyond Fig. 1, see Refs. [9,10] and references therein.

particle are removed, leaving -1 room on each floor. Here, our hotel analogy becomes somewhat awkward, but the minus sign ought to be dropped. (In the actual problem the negative sign implies a reversal of the direction of the magnetic field, but the physics is independent of whether the magnetic field is up or down.) The composite fermions therefore fill the lowest three floors to produce a unique state. Such definite answers are not obtained for even denominator fractions. For example, consider three electrons with eight rooms on each floor ($\nu = \frac{3}{8}$). For the composite fermions with $2p = 2$, we have two rooms on each floor; two electrons occupy the first floor, but the third has a choice of two rooms on the second floor. Such ambiguity is the reason for the absence of plateaus characterized by even denominator fractions. (Note to the advanced reader: our discussion assumes *non*-interacting composite fermions. When the relatively weak interaction between them is included in the theory, secondary new structures, including even denominator FQHE, become a possibility.)

Thus, the numerology of the fractions is explained beautifully. The exactness of the quantization is understood as a consequence of the fact that gaps appear at *precisely* the fractions given by Eq. (12), the right-hand side of which is a combination of whole numbers, hence insensitive to small perturbations.

When one takes the notion of composite fermions seriously, one begins to realize that it has numerous other, at-first-sight-bizarre consequences, the kind you would never believe, or even think of, if you did not know about composite fermions. A large number of such effects have already been verified in experimental and theoretical studies during the last decade and a half (Fig. 4), clinching the case for composite fermions. To date, the existence of four different "flavors" of composite fermions has been established, those carrying two, four, six, and eight flux quanta (Table 1), and many of their properties have been investigated. The relevant scientific papers are too numerous to be listed individually in this short, introductory article; the reader interested in learning more about these developments is referred to Refs. [3,4].

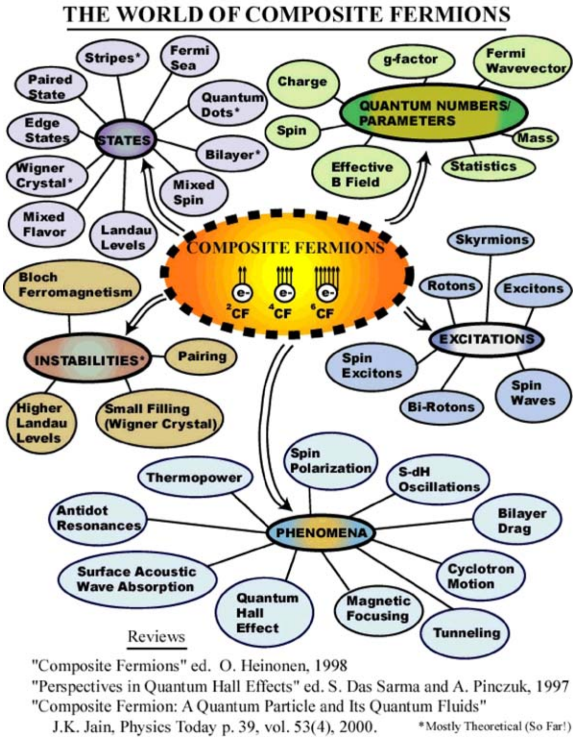

Fig. 4. Various facets of composite fermion physics (*source*: V.W. Scarola, 2001).

7. Microscopic theory

The above considerations show how the qualitative phenomenology of the FQHE is explained in the same language as the IQHE. Can we exploit the analogy further to formulate a microscopic theory of the FQHE? In other words, can we construct microscopic wave functions for n filled shells of composite fermions, which are to be identified with the wave functions of interacting electrons at $v = n/(2pn \pm 1)$, with the help of the *known* wave functions for n filled shells of electrons? That is indeed the case, and the wave functions are given by [7]

$$\Psi_{n/(2pn\pm 1)} = \prod_{j<k} (z_j - z_k)^{2p} \Phi_{\pm n}. \tag{13}$$

Here $z_j = x_j - \mathrm{i}y_j$ denotes the electron coordinates as a complex number, Φ_n is the Slater determinant wave function for n filled shells of electrons, and $\Phi_{-n} = \Phi_n^*$. (Complex conjugation is equivalent to switching the direction of the magnetic field.) These wave functions were inspired by a combination of the analogy between the FQHE and the IQHE discussed above and the seminal wave function invented earlier by Laughlin [8] for an explanation of the $v = 1/m$ FQHE, m odd. Laughlin's wave function, which coincides with $\Psi_{1/(2p+1)}$, acquires the physical meaning of one filled shell of composite fermions ($n = 1$).

The wave functions of Eq. (13) contain no adjustable parameters (for a given filling factor), because they are obtained from the parameter-free filled shell IQHE states at $v = n$. The only thing variational about them is their form, which is dictated by the composite fermion physics; once the form is assumed, the wave function is uniquely fixed. For quantitative studies, they are to be projected into the lowest electronic Landau level, as appropriate in the limit of very high magnetic fields. Even though the resulting wave functions are extremely complicated, it has been possible to deduce numbers from them (often with the help of Monte Carlo). Exact computer experiments on hundreds of systems with different N and M have shown that these wave functions, as well as their numerous extensions, are practically exact representations of the true eigenstates; for example, the eigenenergies are predicted with an accuracy of $\sim 0.1\%$ or better. Such quantitative accuracy is astonishing in view of the lack of any variational freedom in the theory, and proves the formation of composite fermions and the connection between the fractional and the integral effects at a microscopic level. A unique theory has thus materialized for a problem that had, to begin with, an astronomically large number of possibilities.

Eq. (13) brings out the true microscopic meaning of composite fermion: it is the bound state of an electron and $2p$ quantized *vortices*. (The flux quanta of Fig. 2 are a topologically faithful representation of the microscopic vortices.) Due to the Jastrow factor $\prod_{j<k}(z_j - z_k)^{2p}$, every electron sees $2p$ vortices on every other electron. In other words, multiplication by the Jastrow factor binds $2p$ vortices to each electron in $\Phi_{\pm n}$ to convert it into a composite fermion, and the right hand side is interpreted as n filled shells of composite fermions. To take a concrete example, if the arrows (which represent vortices) were removed in Fig. 3, it would represent Φ_3, that is, three filled shells of electrons; multiplication by the Jastrow factor $\prod_{j<k}(z_j - z_k)^4$ endows electrons with arrows to produce $\Psi_{3/13}$.

The microscopic theory explains why composite fermions are produced in the first place, and also why they are weakly interacting. Typically, for fermions, the probability of two fermions approaching one another vanishes as r^2, where r is the distance separating them. In $\Psi_{n/(2pn\pm1)}$ the probability goes as r^{4p+2}, indicating that it is very efficient in keeping electrons away from one another, which is exactly what they want. In other words, electrons transform into composite fermions by dressing themselves with quantized vortices, because that is how they can best avoid one another. The weakness of the interaction between composite fermions follows from the general rule that the interaction between bound states is weaker than that between the constituent particles. (For example, nucleons interact much more feebly than quarks.) Here, most of the Coulomb interaction between the electrons has been consumed in making the composite fermions, which themselves feel only a faint residual interaction. The actual form of the inter-composite fermion interaction is obviously very complicated, but it is altogether negligible for many purposes, and, if needed, can be estimated with the help of the microscopic theory.

The magnetic field reduction can also be understood microscopically. As composite fermions move about carrying their vortices, the Aharonov Bohm phase due to the external magnetic field is partly canceled by the phase generated by the vortices on other composite fermions. Because each vortex produces the same phase as a flux quantum, the net effect is as though the magnetic field were reduced from B to B^*. (The reader may be reminded here of the Meissner effect in superconductors, but that has a different physical origin. There, a cancellation of the magnetic field occurs due to induced macroscopic surface currents that produce, according to the laws of classical electrodynamics, an opposing magnetic field. There are no such currents here. The partial cancellation of the vector potential is a purely quantum mechanical effect sensed only by the composite fermions; an external magnetometer would still read the full magnetic field.)

Note that even though the wave functions in Eq. (13) are very accurate, they are not exact. However, that is not a problem. The aim in condensed matter physics is not to test the starting point of the theory, namely the Schrödinger equation, but to uncover new structures that emerge when many particles behave cooperatively. A simple approximation to the exact solution is often more important than the exact solution itself, because it allows an insight into the *physics* of the problem, bringing out the non-perturbative features that are more generally valid than many details of a specific solution. (Given that the exact solution for the general problem of *three* interacting particles is not yet available, condensed matter systems with many more particles are not amenable to *exact* treatments. The best one can hope for is to find either an approximate solution to the actual problem or an exact solutions to an approximate problem.) To appreciate the significance of what has been accomplished, imagine diagonalizing the problem exactly on a huge computer to obtain the ground state at certain filling factor. The computer would display a long list of numbers, which are the projections of the ground state vector along various directions of the very large dimensional Hilbert space. But it would not tell us what these numbers really mean. Knowing that this complicated wave function is very well approximated by the compact form in Eq. (13) reveals that vortices are bound to electrons, and that the bound states can be interpreted as fermionic particles moving in an effective magnetic field. It is also worth stressing here that the approximate nature of the wave functions has no bearing on the exactness of the quantization of the Hall resistance; this is similar to superconductivity, where the value of the flux quantum, $h/2e$, is an exact consequence of electron pairing, unspoiled by the approximate quality of the BCS wave function.

It must be remembered that what appears simple in terms of composite fermions is a non-trivial, non-perturbative state of electrons. The composite fermion itself is an exceedingly complicated object from the electrons' vantage point, because the quantized vortex, one of its constituents, is a collective entity in which all electrons participate. (Some may find it surprising that composite fermions behave as ordinary fermions to the extent they do.) On the flip side, the good old electron is complex beast when viewed from the reference frame of composite fermions.

To conclude, the FQHE of electrons is an IQHE of composite fermions. The lowest Landau level quantum fluid is a colossal atom of composite fermions, and the FQHE states are inert atoms containing filled shells.

8. Back to basic quantum mechanics

Theory's job is to identify simple underlying principles that allow us to unify, explain, predict and calculate. For the lowest Landau level quantum fluid, nature's generosity in providing with clues has led us to a single principle, the formation of composite fermions, which, with no fine tuning, synthesizes the magnificent phenomenology of the FQHE and numerous other related observations, and at the same time gives an accurate microscopic theory with predictive power. With composite fermions, a host of facts suddenly fall into place, and what earlier seemed mysterious becomes all too natural, even inevitable. At the most fundamental level, the emergence of composite fermions describes the distinct manner in which quantum mechanical phases, in the form of quantized vortices bound to electrons, enter into the physics of the lowest Landau level quantum fluid. The composite fermion theory firmly links the physics of the FQHE and the IQHE, thereby achieving a pleasing unification of the two phenomena.

An unusual feature of this problem is worth noting. As the increasing use of field theory in condensed matter physics indicates, many-body problems are often replaced by simpler "effective" problems that, it is hoped, correctly capture certain qualitative and semi-quantitative features. The inability to do better is hardly surprising in view of the fact that systems with a large number of interacting particles are horribly complex. If anything is surprising, it is that we are able to make any progress at all.

In the case of the lowest Landau level quantum fluid, we are in the happy situation that we have found accurate solutions of the *original* interacting electron problem defined in Eq. (6), the high field limit of Eq. (1). The explanation of this state takes us back to the beautiful simplicity and the raw power of quantum theory, the kind that was in play in atomic physics in the glorious early days of quantum mechanics, when we could take the actual Hamiltonian, determine its eigenfunctions and eigenenergies, and then see with awe and wonder how it all fit so neatly with experiment.

9. There's more ...

I have described above only the most essential aspects of the theory of the fractional quantum Hall effect. There is, of course, a lot more, which can be found in Refs. [3,4]. We have come a long way, but much work lies ahead of us. The theory continues to be further developed, refined and applied to new phenomena. The domain of its validity is being ascertained; for example, there are filling factor regions where the composite fermion fluid is superseded by a different kind of state, for example, a Wigner crystal or a charge density wave. On the experimental front, exciting discoveries are being made with uncanny regularity. Some very recent experimental advances can be found in Refs. [9–22]. If the past is any guide, further exploration of this quantum fluid will continue to delight us with unanticipated revelations.

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