Homework 10 Physics 419

Due: Nov. 12

1. Hamiltonian in rotating coordinates.

A bug is crawling on a turntable rotating around its axis in some unspecified, time-dependent fashion. A physicist standing on the turntable measures the bugs position in terms of coordinates x, y fixed with respect to the turntable, while we (in an inertial frame) measure its position in terms of lab coordinates x_{lab}, y_{lab}. The polar coordinates are $r = r_{lab}$, $\theta = \theta_{lab} - \phi(t)$, where $\phi(t)$ is the angle between the two coordinate systems, which may be changing in a complicated way. The Lagrangian, in mixed coordinates is

$$L = \frac{m}{2}|\mathbf{v}_{lab}|^2 - V(r, \theta).$$

(a) Why do we use the Lab frame velocity $\mathbf{v}_{lab}$ in the Lagrangian rather than $\mathbf{v}$ (turntable frame)?
(b) Substitute the rotating coordinates into the expression for the lab kinetic energy to find the canonically conjugate momenta p_r and p_θ.
(c) Calculate the bug's Hamiltonian in terms of r,θ, p_r, p_θ. Show that, whatever $\phi(t)$ is,

$$H = H_{lab} - \dot{\phi} p_\theta,$$

where H_{lab} is the Hamiltonian if $\dot{\phi} = 0$.
[If you are having trouble, look at the polar coordinates example in the notes and the elevator example we did in class.]

2. droplets in phase space .
Look at the phase portrait for the plane pendulum which appears on page 84 of the Notes. Remember, that should actually be thought of as wrapped around a cylinder. Imagine a small circular droplet of initial conditions around the stable fixed point. Describe what happens to it as time goes on and it is carried by the phase flow. Do the same for a droplet around the unstable fixed point. How is Liouville's theorem relevant to this? You may like to draw pictures.

3. coordinate changes and phase space volume .
The object here is to show that phase space volume is preserved by configuration coordinate changes. That is, if $x^1, \ldots, x^n$ is one set of configuration coordinates and $y^1, \ldots, y^n$ is an alternate set,

$$dy^1 \cdots dy^n dp_{y1} \cdots dp_{yn} = dx^1 \cdots dx^n dp_{x1} \cdots dp_{xn}. \tag{1}$$

The y^1,...,y^n are functions only of x^1,...,x^n, and not of the p_{xi}.
(a). Work out the one degree of freedom case. By definition,

$$p_y = \frac{\partial L}{\partial \dot{y}}\bigg|_y = \frac{\partial L}{\partial \dot{x}}\bigg|_x \frac{\partial \dot{x}}{\partial \dot{y}}\bigg|_y + \frac{\partial L}{\partial x}\bigg|_{\dot{x}} \frac{\partial x}{\partial \dot{y}}\bigg|_y, = p_x \frac{\partial \dot{x}}{\partial \dot{y}}\bigg|_y + \frac{\partial L}{\partial x}\bigg|_{\dot{x}} \frac{\partial x}{\partial \dot{y}}\bigg|_y, \tag{2}$$

where the bars indicate the things which are held constant during the associated differentiation. The first term can be handled by

$$\frac{\partial \dot{x}}{\partial \dot{y}}\bigg|_y = \frac{\partial x}{\partial y}\bigg|_y,$$

which is equation (2.2.5) in the Notes. You should look at the discussion around that equation to be sure that makes sense to you.

Show that the second term in equation (2) vanishes, and then use that to establish equation (1) for this case.

(b). Now try the many degree of freedom case. To do that, you will need to know that the determinant of a block diagonal matrix is the product of the determinants of the blocks:

$$\det \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} = \det \mathbf{A} \cdot \det \mathbf{B}.$$

In this equation, $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{0}$ stand for complete matrices themselves. The other fact you will need is the Jacobian formula for changes of variables:

$$dx'_1\, dx'_2 \cdots dx'_n = |det J| dx_1\, dx_2 \cdots dx_n,$$

where J is the Jacobian matrix

$$J_{ij} = \frac{\partial x'_i}{\partial x_j}.$$

Note that the x's and x''s here are completely general variables, not necessarily related to the x's appearing earlier in the problem. I.e., some of these x's could be taken to be p's.