Homework 2 Physics 419

Due: Sept. 20

1. **index notation practice.** Here is some practice in using the index notation. See MT §1.10-1.12 for help.

   There are two special symbols which occur a lot with the index notation. One is the Kronecker delta, δ_{ij}, which is 1 if $i = j$ and zero otherwise. The other is the Levi-Civita symbol ε_{ijk}. If you imagine putting 1,2 and 3 around a clock face, an even permutation of them is one where they are in clockwise order and an odd permutation is one where they are in counterclockwise order. So, 231 is even and 213 is odd. ε_{ijk} is $+1$ if ijk is an even permutation of 123, -1 if it is an odd permutation and zero otherwise (i.e., if two indices match).
   The ε symbol is very useful when dealing with cross products.

   Terse as the idiom is, we often get tired even of writing the summation symbols, and lapse into the so-called summation convention. Under this convention, any repeated index in the same term (called a dummy index) is summed over the values 1 to 3, unless otherwise indicated. Using the summation convention, we get, for example $\delta_{ii} = 3$, $\nabla \cdot \mathbf{g} = \partial g_i/\partial x^i$, and $\nabla f = \hat{\mathbf{e}}_i \partial f/\partial x^i$. Note that i is not a repeated index in $a_{ik} + b_{ik}$ because the two occurences are in different terms.

   The cross product of **a** and **b** is

   $$(\mathbf{a} \times \mathbf{b})_i = \epsilon_{ijk} a_j b_k.$$

   The indices j and k are summed over. Check that this matches expressions you're familiar with.

   Now, verify the following relations ($\mathbf{x} = \hat{\mathbf{e}}_i x_i$ is a position vector):

   (a) $\delta_{ij}\delta_{jk} = \delta_{ik}$

   (b) $\nabla(\mathbf{x} \cdot \mathbf{x}) = 2\mathbf{x}$

   (c) $\nabla(|\mathbf{x}|) = \mathbf{x}/|\mathbf{x}|$

   (d) $\partial(x^j/|\mathbf{x}|)/\partial x^i = (|\mathbf{x}|^2 \delta_{ij} - x_i x_j)/|\mathbf{x}|^3$

   (e) $\varepsilon_{ijk}\varepsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$, hence $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$

   (f) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$

   (g) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$

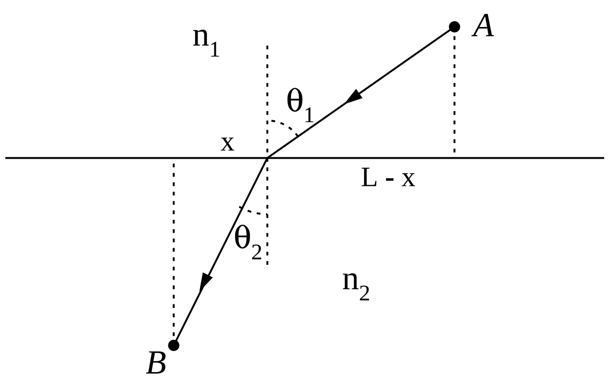
   (h) write $\mathbf{v} \times (\nabla \times \mathbf{A})$ in index notation.

2. **wheels within wheels.** A hollow cylindrical shell of radius a rolls without slipping on the lower half of the inner surface of another hollow cylinder of inner radius A (on the lower half) under a gravitational acceleration g. The larger cylinder is stationary. Determine the Lagrangian, and then the Lagrangian equations of motion. How many degrees of freedom are there?

3. **spherical pendulum** A spherical pendulum is one which is free to swing in any direction, the only constraint being on the distance from the point of support. How many degrees of freedom does a spherical pendulum have? Using spherical coordinates, find the Lagrangian and the equations of motion for a spherical pendulum (mass m) supported by a massless rod of length ℓ (gravitational acceleration g).

4. **shaken not stirred.** The point of support of a plane pendulum (length ℓ) is driven vertically according to $y = a \cos \omega t$. Find the Lagrangian for this pendulum and the equations of motion.

5. **Fermat's Principle and Snell's Law.** According to Fermat's Principle, the path of a light beam between two points is the one which minimizes the transit time. Snell's law states that when light crosses an interface between media with different indices of refraction, the product $n \sin \theta$ is the same on both sides, where θ is the angle between the normal to the interface and the beam (see the figure). Derive Snell's law from Fermat's Principle and the fact that the speed of light is c/n in a medium of refractive index n. You probably want to do this by varying x in the figure so as to minimize the light travel time from A to B.



6. **Fermat's Principle and a conservation law.** Sometimes a continuously varying index of refraction is relevant. A case in point is the reflection of radio waves from the ionosphere. Then it is difficult to use Snell's law since the situation is like a stack of infinitesimally separated interfaces. The total transit time of a ray is

$$T = \int \frac{d\ell}{v} = \frac{1}{c} \int n(z) d\ell,$$

where z is altitude.

(a) For unspecified variation of index of refraction with altitude, $n(z)$, show that the Euler equation leads to

$$\frac{z''}{1 + z'^2} = \frac{dn/dz}{n}.$$

(b) Consider a functional

$$I[y] = \int f(y(x), y'(x))\, dx$$

of a function $y(x)$. This is of the sort we've been considering, except that there is no explicit dependence of f on x. Show that

$$\frac{d}{dx}\left(y'\frac{\partial f}{\partial y'} - f\right) = 0.$$

The quantity in parentheses is independent of x, so this is like a conservation law. This equation is derived in MT, §6.4, but it can be done more directly and cleanly under the stated assumption.

(c) Determine the corresponding quantity for the refraction problem discussed at the beginning of the problem, using the form

$$n(z) = n(0)e^{-z/\lambda}$$

for the refractive index. Find the maximum height of the ray as a function of the original direction of motion $z'(0)$. (It's "launched" from $z = 0$)

(d) Transcribe the result of part (b) into the mechanical context to obtain a genuine conservation law for the case that the Lagrangian is independent of time. For a system of particles interacting via a potential U, but not subject to any outside influences, evaluate the mystery conserved quantity. (Use cartesian coordinates) Does it look familiar?

7. **Charged particle in an electromagnetic field.** The Lorentz force on a charged particle of charge e in an electromagnetic field leads to an equation of motion

$$m\ddot{\mathbf{r}} = e\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$$

in SI units. This force is velocity-dependent and therefore cannot possibly come from a potential $V(\mathbf{r})$. You are familiar with the use of a scalar potential ϕ for the electric field, $\mathbf{E} = -\nabla\phi$, but maybe don't know about vector potentials. It happens that it is possible to express both $\mathbf{E}$ and $\mathbf{B}$ in terms of ϕ and a *vector potential* $\mathbf{A}$ in this way:

$$\begin{aligned} \mathbf{E} &= -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} \\ \mathbf{B} &= \nabla \times \mathbf{A}. \end{aligned}$$

Both ϕ and $\mathbf{A}$ are in general functions of position and time. [The reason $\mathbf{A}$ is called a potential is that the directly measurable quantities are expressed as first derivatives.]

(a) In terms of the vector potential, the magnetic force on the particle can be written as $\mathbf{v} \times (\nabla \times \mathbf{A})$. Compute the x-component of this directly and then determine the y and z components by symmetry. If you prefer, you are welcome to do it all in one blow using the ε_{ijk} symbol.

(b) A Lagrangian for the charged particle in an EM field is

$$L = \frac{m}{2}|\mathbf{v}|^2 + e\left[\mathbf{v} \cdot \mathbf{A}(\mathbf{r}, t) - \phi(\mathbf{r}, t)\right]. \qquad (1)$$

Notice that this does not have the form $T - V$. You are going to determine the corresponding Euler-Lagrange equations of motion. Before getting to that, notice that $\mathbf{A}$ is a function of position and time. So, $d\mathbf{A}/dt$, which is the time derivative of $\mathbf{A}$ along a phase space trajectory, takes the form

$$\frac{dA^i}{dt} = \sum_j \dot{x}^j \frac{\partial A^i}{\partial x^j} + \frac{\partial A^i}{\partial t}.$$

Explain what this equation means.

(c) Now, using the results of parts (a) and (b), work out the Euler-Lagrange equation for the x coordinate of the particle. Obtain those for y and z by a symmetry argument. Does this result agree with the Lorentz force? (It had better). Again, if you feel comfortable with it, you may do all three components at once using the index notation.

(d) (optional) There is an arbitrariness in $\mathbf{A}$ and ϕ known as gauge freedom. A gauge transformation takes the form

$$\begin{aligned} \mathbf{A} &\rightarrow \mathbf{A} + \nabla\chi \\ \phi &\rightarrow \phi - \frac{\partial\chi}{\partial t}, \end{aligned}$$

where χ is an arbitrary smooth function of position and time. Verify that this change does not affect $\mathbf{E}$ or $\mathbf{B}$. How does it alter the Lagrangian? Does that difference change the equations of motion? How does it affect the action for paths with specified endpoints? Do you see a general result about freedom to alter the Lagrangian for a system without affecting the equations of motion?