Homework 3 Physics 419

Due: Sept. 29

1. In a short while, we will be very concerned with the two-body problem. This exercise is meant to give you a running start on it, and also to help your understanding of Lagrange multipliers and constraint forces.

(a) Consider a system of two particles, with masses m_1 and m_2, subject to a completely generic potential $U(\mathbf{r}_1, \mathbf{r}_2)$, and are held a fixed distance d apart by a massless rigid rod.

(b) Characterize the various degrees of freedom of this system. How many are there? Give a geometrical description of the configuration space $\mathcal{Q}$.

(c) Instead of using that configuration space directly, you are going to work with the configuration space $\mathcal{Q}_0$ of two *free* particles, with the constraint function

$$g(\mathbf{x}_1, \mathbf{x}_2) = |\mathbf{x}_1 - \mathbf{x}_2|,$$

and restrict the motion to $g(\mathbf{x}_1, \mathbf{x}_2) = d$ by means of a Lagrange multiplier function $\lambda(t)$. Specify another function that you could use just as well as this g that I have chosen.

(d) Now compute the kinetic energy using Cartesian coordinates on $\mathcal{Q}_0$. Combine this with the potential and the constraint to get the 'pseudo-Lagrangian'

$$\tilde{L} = L + \lambda g.$$

(e) From your pseudo-Lagrangian, determine the equations of motion. Observe that the way λ shows up in these equations makes it clear that it represents a force, or forces. Which way do they point? Is it consistent with Newton's 3rd Law?

(f) Actually, a different set of coordinates on configuration space is often preferable to the ones you've used. These are center of mass $\mathbf{R}$ and relative separation vector, given by

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2.$$

The positions of the two particles relative to the center of mass are denoted by

$$\mathbf{r}'_i = \mathbf{r}_i - \mathbf{R}, \quad i = 1, 2.$$

Express these in terms of the separation vector $\mathbf{r}$. Is it surprising that both of these can be expressed in terms of $\mathbf{r}$? Draw a picture. Write down $\mathbf{r}_1$ and $\mathbf{r}_2$ in terms of $\mathbf{R}$ and $\mathbf{r}$.

(g) Using the results you've just derived, re-express the kinetic energy you obtained in part (d) in terms of $\mathbf{R}$, $\mathbf{r}'_1$ and $\mathbf{r}'_2$. You should find that it splits into one piece depending only on $\dot{\mathbf{R}}$ and another depending on $\dot{\mathbf{r}}'_1$ and $\dot{\mathbf{r}}'_2$. This is a special case of a general result. What is that general result? (see section 1.6 of the Notes or section 9.5 of MT)

You have too many variables. Rewrite the kinetic energy yet again, this time in terms of $\mathbf{R}$ and $\mathbf{r}$. These are your new coordinates on $\mathcal{Q}_0$. Again, there is a separation, one term depending on $\dot{\mathbf{R}}$ and another on $\dot{\mathbf{r}}$. The reduced mass

$$\mu \stackrel{def}{=} \frac{m_1 m_2}{m_1 + m_2}$$

should come in handy.

(h) Work out the Lagrangian equations of motion with these new (center of mass and relative separation) coordinates.

(i) Set $U(\mathbf{r}_1, \mathbf{r}_2) \equiv 0$. Solve your equations of motion for $\lambda(t)$. To do this, you impose by hand the requirement that $|\mathbf{r}|^2 = d^2$ and check what that implies about λ. It is helpful to notice that the constraint implies

$$\frac{1}{2}\frac{d^2}{dt^2}|\mathbf{r}|^2 = |\dot{\mathbf{r}}|^2 + \mathbf{r} \cdot \ddot{\mathbf{r}} = 0.$$

(j) For simplicity, set $m_1 = m_2$ and take $\mathbf{r}$ to be a two-dimensional vector. Evaulate λ in this case. Does it agree with what you would get by old techniques?

2. Noether's Theorem. Beyond the ones we discussed in class, the purely mechanical applications of Noether's Theorem are not plentiful. Here is one, though.

Consider a single particle moving in a potential U with the property that

$$U(\rho, \theta, z) = U(\rho, \theta - 2\pi z/d, 0),$$

(using cylindrical coordinates with $\rho = \sqrt{x^2 + y^2}$ and $\theta + 2\pi$ identified with θ.)

(a) Write down the kinetic energy of your particle in cylindrical coordinates, and then the Lagrangian.

(b) The invariance of the potential displayed in part (a) can also be written in differential form as $dU/ds = 0$, with

$$d\theta/ds = 1, \quad dz/ds = ??, \quad d\rho/ds = ???.$$

Fill in the '??' and '???'. Describe in geometrical terms the motion of points as s is increased from zero.

(c) The kinetic energy is also invariant under this group of transformations. Explain how you know this to be so. Explicit computation is not necessary or desirable, but if you have to resort to that, do so.

(d) Using Noether's Theorem, compute the conserved quantity which is associated with the group of transformations identified in the previous part.

(e) Rewrite your conserved quantity in terms of a sum of a component of angular momentum and a component of linear momentum.

(f) Imagine some motions consistent with your conservation law.