

Due: Oct. 4

1. **imaginary exponents.** By decomposing the sine and cosine functions into

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}, \quad \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2},$$

prove the double angle formulas

$$\sin(2\theta) = 2 \sin \theta \cos \theta, \quad \cos(2\theta) = \cos^2 \theta - \sin^2 \theta.$$

2. **Some observation.** Find two examples of oscillatory or resonant phenomena in your daily surroundings. Comment on such aspects as damping, whether there are multiple degrees of freedom involved so that the vibrations can have different “shapes” as well as amplitudes, whether you think a linear approximation is reasonable, etc.

3. **A toppling pencil.** Suppose you balance a new, sharpened, pencil on its tip almost perfectly. But not quite. It is tipped by one minute of arc ($1/60^{\text{th}}$ of a degree). Estimate how long it will take to fall over.

- (a) As a preliminary step in doing this problem, calculate the kinetic energy of a thin rod (length L , mass M) rotating with angular velocity ω about an axis passing through the center of the rod and perpendicular to it. Do this without explicitly using the concept of moment of inertia. Slice the rod up into little bits of length dl having mass $(M/L)dl$, multiply by $v^2/2$ (which depends on the position of the slice), and adding it all up, i.e., integrate. You should get $T = (ML^2/24)\omega^2$. Now recognize this as $I\omega^2/2$, and that this is the usual procedure for calculating moment of inertia.
- (b) Getting back to the pencil, assume the center of mass is halfway along the length (close enough) and compute the Lagrangian, being careful to take into account both the motion of the center of mass and rotation of the pencil about the center of mass. Linearize this Lagrangian near the equilibrium. That involves only the potential energy. At this stage you have an easy problem. The initial velocity is zero. You don't actually need to know the mass of the pencil. For ‘fallen over,’ take the condition $\theta = \pi/2$, corresponding to the pencil being horizontal.
- (c) Does your answer seem reasonable, or do you think environmental disturbances are important for this problem?

4. **wheels revisited.** Recall the ‘wheels within wheels’ problem on homework 2. (See homework or solutions to refresh your memory) The equation of motion was

$$\ddot{\theta} + \frac{g}{2(A-a)} \sin \theta = 0.$$

- (a) Linearize this equation about the equilibrium position $\theta = 0$ (small cylinder at the bottom of the larger one). Find the frequency of small oscillations about this position.
- (b) In some respects, the situation is much like a simple pendulum of length $A - a$, since the centers of mass travel on circles of that radius in both cases. However, the frequency for the rolling wheel is lower by a factor of $\sqrt{2}$. Give an argument for why you expect the frequency to be lower, using such concepts as force, energy and work.
- 5. Bead on a bent wire.** About this time you might get the idea that I have a thing for pendulums and beads. Well, at least it's better than Atwood machines.
- (a) A bead slides without friction on a wire which is bent into an upward pointing, more or less 'U' shape, specified by $z(x)$, with $z(0) = z'(0) = 0$. (So, it lies in the x - z plane.) Compute the frequency of small oscillations about the equilibrium, using x as your coordinate. (express in terms of derivatives of z with respect to x)
- (b) (optional) The method of the previous part made the potential energy simple since it was just $U = mgz(x)$. The kinetic energy was complicated, though. If you want the kinetic energy to look simple, you would use arc length s as your coordinate. However, an expression for U in terms of s is hard to come by. If you only want the linearization around the equilibrium, it does not matter, since you only need derivatives of U . Redo part (a) using s as your coordinate. Express derivatives with respect to s as derivatives with respect to x by use of the chain rule.
- 6. Another Green function.** Construct the Green function for the equation

$$\ddot{y} + k\dot{y} = f(t) \tag{1}$$

in two different ways.

- (a) First, use the Green function G_{dsho} for the underdamped oscillator (eq. 3.7.32 in the Notes). Since the equation (1) does not have a y term, you need to take the limit $\omega_0 \rightarrow 0$. But you must also take Q to zero in such a way that $k = \omega_0/Q$ remains fixed.
- (b) Now find the solutions of the homogeneous equation by setting $f(t) = 0$.
- (c) From the solutions of the previous part, construct $G(t)$ by finding the combination which has the desired properties: $G(t) = 0$ for $t < 0$ and $G'(0+) = 1$. ('0+' denotes the limit as t comes down to zero from the positive direction)