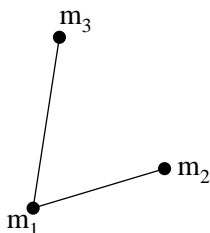


Due: Oct. 15

1. Practice with vector derivatives. Suppose three masses m_1 , m_2 and m_3 are subject to a potential

$$U(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = k[(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{r}_3 - \mathbf{r}_1)]^2.$$

An interesting thing about this potential is that it depends upon the angle between the vectors $\mathbf{r}_{21} = \mathbf{r}_2 - \mathbf{r}_1$ and $\mathbf{r}_{31} = \mathbf{r}_3 - \mathbf{r}_1$. Such dependences arise (for example) in potentials used to model the vibrations of molecules because the chemical bonds prefer to be at certain angles with respect to each other.



The force on particle 1 is

$$\mathbf{F}_1 = -\frac{\partial U}{\partial \mathbf{r}_1} = -\frac{\partial U}{\partial x_1} \hat{\mathbf{e}}_x - \frac{\partial U}{\partial y_1} \hat{\mathbf{e}}_y - \frac{\partial U}{\partial z_1} \hat{\mathbf{e}}_z.$$

This is precisely the gradient of U if \mathbf{r}_2 and \mathbf{r}_3 are thought of as constant vectors, with only \mathbf{r}_1 variable. But, the potential above is a function of \mathbf{r}_1 , \mathbf{r}_2 and \mathbf{r}_3 only in the combinations \mathbf{r}_{21} and \mathbf{r}_{31} . And, the effect on U of shifting \mathbf{r}_1 by a small vector \mathbf{a} with \mathbf{r}_2 and \mathbf{r}_3 held fixed is exactly the same as shifting \mathbf{r}_{21} and \mathbf{r}_{31} by $-\mathbf{a}$. So, considering U as a function of \mathbf{r}_{21} and \mathbf{r}_{31} ,

$$\frac{\partial U}{\partial \mathbf{r}_1} = -\frac{\partial U}{\partial \mathbf{r}_{21}} - \frac{\partial U}{\partial \mathbf{r}_{31}}.$$

That is a sort of vector chain rule, where we've made several applications of the regular chain rule all at once. Similar results hold for the derivatives with respect to \mathbf{r}_2 and \mathbf{r}_3 . As an example,

$$\frac{\partial}{\partial \mathbf{r}} |\mathbf{r}|^2 = 2\mathbf{r} \Rightarrow \frac{\partial}{\partial \mathbf{r}_1} |\mathbf{r}_2 - \mathbf{r}_1|^2 = -2(\mathbf{r}_2 - \mathbf{r}_1).$$

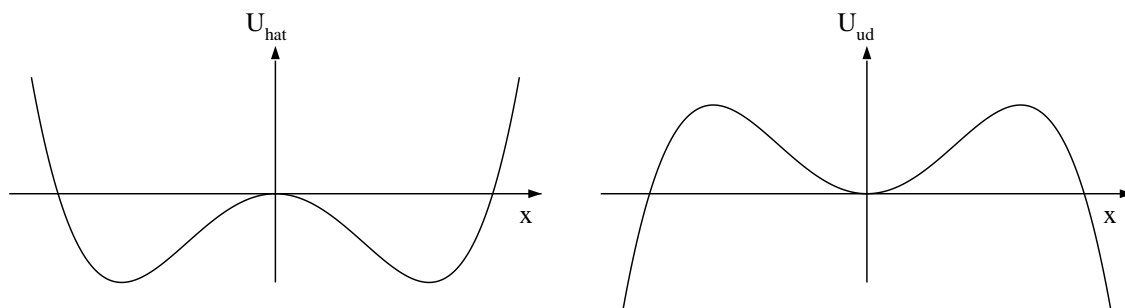
(a) The minimum of the potential $U(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ is zero. Describe the set of configurations on which this minimum is achieved.

(b) Compute the forces on all three particles as a function of the position vectors.

2. Mexican hat potential. Study the motion of a particle in the ‘Mexican hat potential’ given by (U_0 is a unit of energy and a of length)

$$\frac{U_{\text{hat}}(x)}{U_0} = -\left(\frac{x}{a}\right)^2 + \frac{1}{2}\left(\frac{x}{a}\right)^4,$$

and the upside down hat potential $U_{\text{ud}}(x) = -U_{\text{hat}}(x)$. These are shown in the figure.



(a) Sketch phase portraits for a particle moving in each of these potentials. Identify stable (elliptic) and unstable (hyperbolic) fixed points.

(b) If we use a as a unit of length and $t_0 = a/(mU_0)^{1/2}$ as a unit of time, so that $\bar{x} = x/a$, $\bar{t} = t/t_0$, the equation of motion for a mass m particle in the upside-down hat U_{ud} is

$$\frac{d^2\bar{x}}{d\bar{t}^2} + 2\bar{x}(1 - \bar{x}^2) = 0. \quad (1)$$

Show that this is correct.

N.B. In all remaining parts of this problem, we will use these units implicitly, so I won't write the overbars any more.

(c) The unstable fixed points are at $x = \pm 1$. From your phase portrait, you know there is an orbit coming from $x = -1$ in the distant past ($t \rightarrow -\infty$) and going to $x = 1$ in the distant future. In this part you will work out the asymptotic parts of that trajectory and in the next part verify the exact solution.

Linearize the equation of motion (1) about $x = -1$ by defining $y = x + 1$ and expanding the right hand side of equation (1) to first order in y . It is crucial to notice that $d^2y/dt^2 = d^2x/dt^2$. Rewrite $1 - x^2$ as $(1 - x)(1 + x) = (2 - y)y$, and throw away the y^2 . If you're having trouble making the linearization, study the pendulum example near the beginning of chapter 3 in the Notes.

Find the solutions for which $x \rightarrow -1$ as $t \rightarrow -\infty$, and the particle is always moving to the right (so x and y are increasing). You should get

$$x(t) = -1 + y_0 e^{2t}.$$

y_0 is determined from the position of the particle at time $t = 0$: $x(0) = -1 + y_0$. We want the particle to reach the center at time zero, so set $x(0) = 0$.

(d) Make a similar linearization about the right-hand unstable fixed point $x = 1$, and find the solutions for which $x \rightarrow 1$ from the left as $t \rightarrow \infty$. Again, fix the constant by requiring $x(0) = 0$.

(e) In fact, the trajectory which connects the two fixed points and goes through $x = 0$ at time zero can be written down in simple closed form. It is

$$x(t) = \tanh t. \quad (2)$$

\tanh is the hyperbolic tangent, given by

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$

The hyperbolic sine and cosine satisfy many relations analogous to those for the ordinary sine and cosine, except for some changes of sign. For instance,

$$\cosh^2 z - \sinh^2 z = 1,$$

$$\frac{d}{dz} \sinh z = \cosh z, \quad \frac{d}{dz} \cosh z = \sinh z.$$

You would do well to verify these, but it does not need to be handed in.

With the aid of these identities, check that equation (2) really is a solution to the equation of motion.

(f) Use the relation

$$\frac{1}{1+x} = 1 - x + \dots, \quad (x < 1) \quad (3)$$

to verify that when $t \rightarrow \pm\infty$, the exact solution matches onto the general solutions you found in parts (c,d) for the linearized problems.

Notice that it does not match the solutions found in part (c) and (d) which went through $x = 0$ at time $t = 0$. Instead, the solution it matches onto for $t \rightarrow -\infty$ is one which goes through $x = 0$ earlier than $t = 0$. Explain the sign of this discrepancy.

[Equation (3) is the first two terms of the identity you used on problem 6 of the last homework.]