Homework 6 Physics 419

Due: Oct. 18

1. similarity transformations.

Suppose a particle moves in a potential of the form

$$U(\mathbf{r}) = \frac{k}{m} r^{\alpha}.$$

The equation of motion arising from this is

$$\frac{d^2\mathbf{r}}{dt^2} + \alpha k r^{\alpha-1}\hat{\mathbf{e}}_r = 0. \tag{1}$$

(a) Define a rescaled position and time by

$$\mathbf{R} = \lambda \mathbf{r}, \quad \tau = \lambda^z t. \tag{2}$$

λ is a scaling parameter.

Find the value of the exponent z which makes $\mathbf{R}(\tau)$ satisfy the same equation of motion as $\mathbf{r}(t)$, i.e. so that

$$\frac{d^2\mathbf{R}}{d\tau^2} + \alpha k |\mathbf{R}|^{\alpha-1}\hat{\mathbf{e}}_r = 0. \tag{3}$$

z depends only on α.

(b) Deduce from this result that the period of a simple harmonic oscillator is independent of its amplitude.

(c) Interpret the result of part (a) in words.

(d) Deduce Kepler's third law without the constants i.e., show that if an orbit is scaled up by λ, the period increases by a factor $\lambda^{3/2}$.

2. $|\mathbf{r}|^2$ potential.

A particle of mass m moves in two dimensions in a potential $U(\mathbf{r}) = k|\mathbf{r}|^2/2$. Use x and y as coordinates and solve directly for $x(t)$ and $y(t)$, with initial conditions $\mathbf{r}(0)$, $\dot{\mathbf{r}}(0)$. Is your result consistent with the previous problem? Show that the trajectories are ellipses.

3. weighing the sun and the earth.

Kepler's third law can be used to find the masses of the sun and the earth. The gravitational force between bodies of masses M and m is GMm/r^2, so that the k we have been using is GMm in this case.

(a) Since the earth is much much less massive than the sun, μ for the orbital motion of the earth about the sun is not much different from the earth's mass itself. The universal gravitational constant is $G = 6.67 \times 10^{-11}$ N m^2/kg^2, and the mean radius (semimajor axis a) of the earth's orbit is 1.49 $\times 10^8$ km. Find the mass of the sun.

(b) The orbital period of the moon about the earth is a lunar month, 27.3 days, and the mean radius of its orbit is 3.8 $\times 10^5$ km. Find the mass of the earth. Given that the radius of the earth is 6380 km, find the mean density of the earth.