Homework 7 Physics 419

Due: Oct. 22

1. Baby Bert.

Bertrand's theorem asserts that the only central potential for which all bounded orbits are closed are the power law potentials $U = kr^\alpha$ with $\alpha = -1$ and 2. (k being negative in the former case and positive in the latter). The handout I gave you shows that there are no other power law potentials with generically closed orbits. But what of other central-force laws? Your task is to eliminate everything except power laws or a logarithmic potential. Here's how.

(a) For dealing with orbits, it is natural to work with θ and $u = 1/r$ rather than r and t. We showed in class that the 'equation of motion' for u is

$$\frac{d^2u}{d\theta^2} = -\frac{\mu}{\ell^2}\frac{dV_{\text{eff}}}{du} = -u - \frac{\mu}{\ell^2}\frac{dV}{du}.$$

V is completely unspecified at this point except that it is a function of r only (or u). The condition for a circular orbit is $d^2u/d\theta^2 = 0$. Directly from the equation of motion, express this as a relation between u, ℓ and dV/du. (I mention ℓ and not μ because the latter is fixed, but ℓ is variable.)

(b) Check that what you've found is really nothing other than the familiar equation

$$\text{centripetal force} = \frac{mv^2}{r}.$$

That requires writing dV/du in terms of dV/dr, since $\mathbf{F} = -(dV/dr)\hat{\mathbf{e}}_r$.

(c) Now find the frequency (squared) of small oscillations of u about the value $u_0(\ell)$ corresponding to the circular orbit you worked out in part (a). Do that by expanding the potential to second order in the deviation from u_0. This is not a frequency in the usual sense since the independent variable is θ and not t, but that language is convenient as long as we don't become confused. The answer contains a second derivative of V.

(d) If there is any hope at all of having all bounded orbits closing, the frequency you found in part (b) must be independent of the radius u_0 around which you perturbed. For the reasoning behind that assertion, see the Bertrand theorem handout. That is perhaps the most important part. If you insist that the frequency squared is independent of u, the equation you derived in part (b) is a differential equation for V as a function of u, with which you can make progress. Use it to show that V must be proportional to either a power of r or the logarithm of r. You will need to integrate twice.

2. Laplace-Runge-Lenz vector.

This exercise is not to be handed in. Please try to work it through anyway.

(a) Review for yourself the arguments which lead us to conclude that energy, total linear momentum and angular momentum about the center of mass are conserved for two masses interacting by a central force.

(b) Because of those constants of the motion, the problem is equivalent to a single mass of mass $\mu = m_1m_2/(m_1 + m_2)$ moving in a plane (position $\mathbf{r}$) under the influence of a potential $V(r)$. We took advantage of that to work in polar coordinates. The vector form of the equation of motion for $\mathbf{r}$ is also useful. Write it down. This is really Physics 201: what is the force vector on a particle moving in a potential?

(c) A special feature of the Kepler problem ($U(\mathbf{r}) = -k/r$) is that the orbits are closed. Actually, this fact is related to yet another conserved quantity,

$$\mathbf{J} = \dot{\mathbf{r}} \times \mathbf{L} - \frac{k}{r}\mathbf{r},$$

which is called the Laplace-Runge-Lenz vector. Expand out the angular momentum to express the first term $\dot{\mathbf{r}} \times \mathbf{L}$ in terms of position and velocity. Now show that $\mathbf{J}$ is conserved, i.e. that $d\mathbf{J}/dt = 0$. To do that, use your equation of motion from part (b), and the triple cross-product identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$.

(d) The fact that $\mathbf{J}$ is constant gives another derivation of elliptical orbits. Orient $\mathbf{J}$ along $\hat{\mathbf{e}}_x$ and compute ($J = |\mathbf{J}|$)

$$\mathbf{r} \cdot \mathbf{J} = J\, r\, \cos\theta.$$

The relation $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$ is useful for the first term, $\mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{L})$, which will turn out to be ℓ^2/μ.

This does not determine J yet. Compare your result to the orbital equation we derived in class and find $J = \epsilon k$. (That's the sleazy way out, but you're tired by now.)