1. As usual, decompose the weight Mg of the box into components along and normal to the incline. The normal component

$$N = Mg\cos\theta$$

is cancelled by a force exerted by the incline itself. So we only worry about the forces along the incline:

$$Mg\sin\theta + F_{\text{friction}} - \tau$$

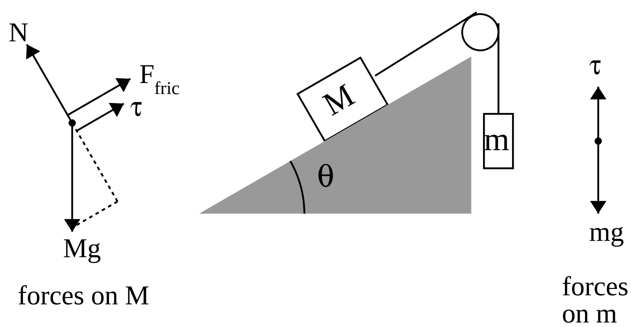
where τ is the tension in the string and the frictional force has magnitude

$$|F_{\text{friction}}| = \mu_k Mg\cos\theta.$$

Its direction depends upon which way the box is sliding. On the hanging mass m, the net downward force is

$$mg - \tau.$$

In order for things to move at constant speed, the net force on each mass must be zero:



$$\tau = mg,$$

$$Mg\sin\theta \pm \mu_k Mg\cos\theta - mg = 0.$$

Isolating $\cos\theta$ in the last equation and squaring,

$$\left(\sin\theta - \frac{m}{M}\right)^2 = \mu_k^2\cos^2\theta = (1-\sin^2\theta)\mu_k^2.$$

Juggling around,

$$(1+\mu_k^2)\sin^2\theta - 2\frac{m}{M}\sin\theta + \left(\frac{m}{M}\right)^2 - \mu_k^2 = 0$$

The solutions for $\sin\theta$ are

$$\sin\theta = \frac{m/M \pm \left[1+\mu_k^2 - (m/M)^2\right]^{1/2}\mu_k}{1+\mu_k^2}.$$

Notice that steady sliding cannot be acheived for $m/M > 1+\mu_k^2$.

2. Denote the mass of the car by M, its initial velocity by v_0 and its displacement downhill from the beginning of the skid by x. As in problem 1, we only need to worry about force components along the incline. There is gravity and friction, giving a net force

$$F_{\text{net}} = Mg\sin\theta - Mg\mu_s\cos\theta = Mg(\sin\theta - \mu_s\cos\theta).$$

This is a constant, so

$$\frac{dv}{dt} = a = \frac{F_{\rm net}}{M}$$

is immediately integrated to yield

$$v(t) - v_0 = at$$

$$x(t) = v_0 t + \frac{1}{2} a t^2$$

The car comes to rest after time T, having slid a distance L. Putting $v(T) = 0$ in the above,

$$T = -a v_0$$

$$L = -\frac{a}{2} v_0^2 = \frac{v_0^2/2g}{\mu_s \cos\theta - \sin\theta}$$

3. (a) Choose coordinates with the origin at the position of firing and denote the muzzle velocity by v_0. Then the equations of motion are

$$\begin{aligned} \ddot{y} &= -g \\ \ddot{x} &= 0 \end{aligned}$$

with initial conditions

$$\begin{aligned} \dot{y}(0) &= v_0 \sin\theta \\ \dot{x}(0) &= 0 \cos\theta. \end{aligned}$$

Since this is a situation of constant acceleration again, these are easily integrated to

$$\begin{aligned} y(t) &= (v_0 \sin\theta) t - \frac{1}{2} g t^2 \\ x(t) &= (v_0 \cos\theta) t. \end{aligned}$$

The distance $R(t)$ of the projectile from the gun is given by

$$R(t)^2 = x(t)^2 + y(t)^2 = v_0^2 t^2 - v_0 g \sin\theta t^3 + \frac{g^2}{4} t^4.$$

We know that for shallow firing angles, $\theta < \theta_c$, R is always increasing (until it hits the ground, but at least it never decreases!). That means that $dR/dt > 0$. For $\theta > \theta_c$, there is a point at which R stops increasing and then decreases. At that point, dR/dt crosses from positive to negative, i.e., goes through zero. The condition that there is a turnaround point is therefore that

$$\frac{dR}{dt} = 0$$

has a solution. Notice that R^2 is increasing or decreasing if R is, and that the expression for $R(t)^2$ is simpler so we will solve instead

$$\frac{d}{dt} R(t)^2 = 0.$$

(This does introduce a spurious solution at $R = 0$, but we'll throw that out) So,

$$\frac{d}{dt} R^2 = \left[2v_0^2 - 3v_0 g \sin\theta t + g^2 t^2\right] t = 0.$$

Ignoring the fake solution $t = 0$, and solving the quadratic equation,

$$t = \frac{3v_0}{2g} \sin\theta \pm \left[\frac{9}{4} \sin^2\theta - 2\right]^{1/2} \frac{v_0}{g}.$$

These solutions make sense only if the expression inside the square root is positive, so real solutions exist only for

$$\sin^2\theta \geq \frac{8}{9}.$$

Thus,

$$\sin\theta_c = \frac{2\sqrt{2}}{3} \quad \Leftrightarrow \quad \theta_c = 70.53°.$$

(b) You notice the curious fact that the critical angle is completely independent of both g and v_0. One way to have anticipated this is to note that an angle is dimensionless. g and v_0 are the only two dimensionful parameters in the problem and there is no way to combine them into a dimensionless combination. So it was really inevitable.

To get a bit more insight into this phenomenon, we find time and distance scales (in other words, *units*) which are dictated by the problem itself:

$$t_0 = \frac{v_0}{g}, \qquad \ell_0 = \frac{v_0^2}{g}, \qquad g = \frac{\ell_0}{t_0^2}.$$

Measuring in these units, we have dimensionless coordinates and time

$$\overline{x} = \frac{x}{\ell_0}, \qquad \overline{y} = \frac{y}{\ell_0}, \qquad \overline{t} = \frac{t}{t_0}.$$

The equations of motion and initial conditions are rewritten as

$$\begin{aligned} \frac{d^2\overline{y}}{d\overline{t}^2} &= -1 \\ \frac{d^2\overline{x}}{d\overline{x}^2} &= 0 \end{aligned}$$

and

$$\begin{aligned} \left.\frac{d\overline{y}}{d\overline{t}}\right|_{t=0} &= \sin\theta \\ \left.\frac{d\overline{x}}{d\overline{t}}\right|_{t=0} &= \cos\theta. \end{aligned}$$

These equations are now universal, depending only upon the firing angle. For fixed θ, we can change the time and distance scale, which means that the overall size of the trajectory and the time required to traverse it can be altered by changing g and v_0. But the *shape* does not change, and the question the problem posed was only concerned with the shape of the trajectory.