

1. (a) The string exerts a force \mathbf{F} on the rock directed toward the hole in the center of the cylinder. Therefore, the torque about the axis of the cylinder $\mathbf{r} \times \mathbf{F}$ is zero. Since the time rate of change of angular momentum is given by the torque, the angular momentum of the rock about that axis is constant.

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = r(mr\dot{\theta})\hat{\mathbf{e}}_z$$

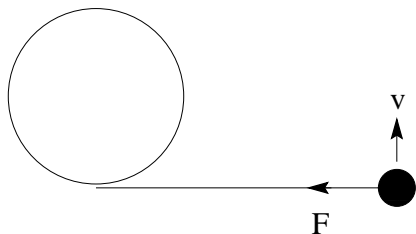
where z is along the axis of the cylinder. Note that although the rock has a radial component of velocity, it does not contribute to the cross product. The condition of the problem, $r = r_0 - ut$ then implies

$$(r_0 - ut)^2\dot{\theta} = L/m = \text{const.}$$

Equivalently

$$\dot{\theta} = \frac{L}{(r_0 - ut)^2}.$$

- (b) Clearly the torque about the center of the spool is not zero here because \mathbf{r} and \mathbf{F} are not parallel. The crucial observation is to note that the rock is instantaneously rotating about the point at which the string leaves the spool. A moment later it is rotating about a different point, but that doesn't matter. That means that the velocity is perpendicular to the force. Since the rate at which the force does work on the rock is $\mathbf{F} \cdot \mathbf{v} = 0$, the kinetic energy is conserved (there is no potential energy anywhere in the problem to worry about).



- (c) Denoting the angular position of the point at which the string leaves the spool by θ ,

$$r(t) = \text{freelengthofstring} = r(0) - R\theta(t).$$

The location of that point is

$$R \cos \theta \hat{\mathbf{e}}_x + R \sin \theta \hat{\mathbf{e}}_y.$$

Since the string is tangent to the spool at that point,

$$\begin{aligned} \mathbf{r} &= R \cos \theta \hat{\mathbf{e}}_x + R \sin \theta \hat{\mathbf{e}}_y + [-r(t) \sin \theta] \hat{\mathbf{e}}_x + r(t) \cos \theta \hat{\mathbf{e}}_y \\ &= (R \cos \theta - r(t) \sin \theta) \hat{\mathbf{e}}_x + (R \sin \theta + r(t) \cos \theta) \hat{\mathbf{e}}_y. \end{aligned} \quad (1)$$

Differentiating with respect to time,

$$\begin{aligned} |\mathbf{v}|^2 &= (-R \sin \theta + r(t) \cos \theta - R \sin \theta)^2 \dot{\theta}^2 + (R \cos \theta - r(t) \sin \theta - R \cos \theta)^2 \dot{\theta}^2 \\ &= r(t)^2 \dot{\theta}^2. \end{aligned}$$

By part (b), this is a constant, and since $\dot{r} = -R\dot{\theta}$,

$$|\mathbf{v}|^2 = \left(\frac{r\dot{r}}{R} \right)^2 = \text{constant}.$$

So, $r dr = c dt$ for some constant c , leading to

$$r = r(0) - bt^{1/2},$$

for a positive constant b , which can be related to the velocity.

2. Divide \mathcal{S} into \mathcal{S}_1 and \mathcal{S}_2 , so that

$$M = \sum_{\alpha \in \mathcal{S}} m_{\alpha} = M_1 + M_2.$$

The centers of mass of the subsystems are given by

$$\mathbf{R}_1 = \frac{1}{M_1} \sum_{\alpha \in \mathcal{S}_1} m_{\alpha} \mathbf{r}_{\alpha}, \quad \mathbf{R}_2 = \frac{1}{M_2} \sum_{\alpha \in \mathcal{S}_2} m_{\alpha} \mathbf{r}_{\alpha}. \quad (2)$$

Now replacing \mathcal{S}_1 by a mass M_1 at \mathbf{R}_1 and similarly with subsystem 2, the center of mass of this new two particle system is

$$\mathbf{R}' = \frac{1}{M_1 + M_2} (M_1 \mathbf{R}_1 + M_2 \mathbf{R}_2),$$

which upon simply substituting from equation (2) becomes

$$\begin{aligned} \mathbf{R}' &= \frac{1}{M} \left(\sum_{\alpha \in \mathcal{S}_1} m_{\alpha} \mathbf{r}_{\alpha} + \sum_{\alpha \in \mathcal{S}_2} m_{\alpha} \mathbf{r}_{\alpha} \right) \\ &= \frac{1}{M} \sum_{\alpha \in \mathcal{S}} m_{\alpha} \mathbf{r}_{\alpha}. \end{aligned}$$

This is precisely the definition of the center of mass of the original system \mathcal{S} , so the procedure is legitimate.

3. With exhaust speed u and external force $F_{\text{ext}} = -mg$ (directed downward), and a lander velocity of zero (it's hovering!), a formula presented in class

$$F_{\text{ext}} = m\dot{v} + \dot{m}u$$

becomes

$$-mg = \dot{m}u.$$

Rearranging

$$\frac{dm}{m} = -\frac{g}{u} dt,$$

which integrates to

$$\ln\left(\frac{m}{m_0}\right) = -\frac{g}{u}t$$

where m_0 is the original mass. Plugging in the indicated values for u , g and $m/m_0 = 0.85$,

$$t_{\text{hover}} = 249 \text{ s.}$$

4. Begin by decomposing the position vector of each particle in the system into the center of mass position \mathbf{R}_{CM} and a position relative to the CM. Then by differentiating, a similar formula arises for the velocities. Together,

$$\begin{aligned}\mathbf{r}_\alpha &= \mathbf{R}_{\text{CM}} + \mathbf{r}'_\alpha \\ \mathbf{v}_\alpha &= \mathbf{V}_{\text{CM}} + \mathbf{v}'_\alpha\end{aligned}$$

Simply inserting these into the definition of angular momentum, expanding out and rearranging terms,

$$\begin{aligned}\mathbf{L} &= \sum_\alpha \mathbf{r} \times \mathbf{p} \\ &= \sum_\alpha m_\alpha \{(\mathbf{R}_{\text{CM}} + \mathbf{r}'_\alpha) \times (\mathbf{V}_{\text{CM}} + \mathbf{v}'_\alpha)\} \\ &= \sum_\alpha m_\alpha [\mathbf{R}_{\text{CM}} \times \mathbf{V}_{\text{CM}} + \mathbf{R}_{\text{CM}} \times \mathbf{v}'_\alpha + \mathbf{r}'_\alpha \times \mathbf{V}_{\text{CM}} + \mathbf{r}'_\alpha \times \mathbf{v}'_\alpha] \\ &= \mathbf{R}_{\text{CM}} \times M\mathbf{V}_{\text{CM}} + \mathbf{R}_{\text{CM}} \times \sum_\alpha m_\alpha \mathbf{v}'_\alpha + \left(\sum_\alpha m_\alpha \mathbf{r}'_\alpha\right) \times \mathbf{V}_{\text{CM}} + \sum_\alpha \mathbf{r}'_\alpha \times \mathbf{p}'_\alpha\end{aligned}\quad (3)$$

Now, from the definition of CM,

$$\begin{aligned}M\mathbf{R}_{\text{CM}} &= \sum_\alpha m_\alpha \mathbf{r}_\alpha \\ &= \sum_\alpha m_\alpha (\mathbf{R}_{\text{CM}} + \mathbf{r}'_\alpha) \\ &= \left(\sum_\alpha m_\alpha\right) \mathbf{R}_{\text{CM}} + \sum_\alpha m_\alpha \mathbf{r}'_\alpha.\end{aligned}$$

Thus, we obtain the key result

$$\sum_\alpha m_\alpha \mathbf{r}'_\alpha = 0.$$

From the right point of view, this is really obvious. Just compute the center of mass with the origin of your coordinate system at the center of mass! Differentiating with respect to time, $\sum_\alpha m_\alpha \mathbf{v}'_\alpha = 0$ as well. So, the second and third terms in equation (3) vanish, leaving

$$\mathbf{L} = \mathbf{R}_{\text{CM}} \times M\mathbf{V}_{\text{CM}} + \sum_\alpha \mathbf{r}'_\alpha \times \mathbf{p}'_\alpha,$$

which is equation (1.4.25) of the notes. Now, going back to $\mathbf{L} = \sum_{\alpha} \mathbf{r}_{\alpha} \times \mathbf{p}_{\alpha}$, we will show that

$$\frac{d\mathbf{L}}{dt} = \sum_{\alpha} \mathbf{N}_{\alpha}^{\text{ext}}.$$

For, differentiating, and using Newton's 2nd Law,

$$\frac{d\mathbf{L}}{dt} = \sum_{\alpha} (\mathbf{v}_{\alpha} \times \mathbf{p}_{\alpha} + \mathbf{r}_{\alpha} \times \dot{\mathbf{p}}_{\alpha}) = \sum_{\alpha} \mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha},$$

since $\mathbf{v}_{\alpha} \times \mathbf{p}_{\alpha} = 0$. Decomposing the force into external and internal parts,

$$\frac{d\mathbf{L}}{dt} = \sum_{\alpha} \mathbf{r}_{\alpha} \times \mathbf{F}_{\alpha}^{\text{ext}} + \sum_{\alpha, \beta} \mathbf{r}_{\alpha} \times \mathbf{f}_{\alpha\beta}. \quad (4)$$

The task is therefore to show that the second sum on the right hand side here vanishes.

By the strong form of Newton's 3rd Law, $\mathbf{f}_{\alpha\beta}$ acts along $\mathbf{r}_{\alpha\beta} = \mathbf{r}_{\alpha} - \mathbf{r}_{\beta}$. Thus, $\mathbf{r}_{\alpha\beta} \times \mathbf{f}_{\alpha\beta} = 0$ and

$$\mathbf{r}_{\alpha} \times \mathbf{f}_{\alpha\beta} = (\mathbf{r}_{\alpha} - \mathbf{r}_{\alpha\beta}) \times \mathbf{f}_{\alpha\beta} = \mathbf{r}_{\beta} \times \mathbf{f}_{\alpha\beta} = -\mathbf{r}_{\beta} \times \mathbf{f}_{\beta\alpha},$$

the last equality arising from the weak form of Newton's 3rd Law, $\mathbf{f}_{\beta\alpha} = -\mathbf{f}_{\alpha\beta}$. Substituting back into the last sum occurring in Eq. (4),

$$\sum_{\alpha, \beta} \mathbf{r}_{\alpha} \times \mathbf{f}_{\alpha\beta} = \sum_{\alpha, \beta} -\mathbf{r}_{\beta} \times \mathbf{f}_{\beta\alpha}.$$

Since α and β are dummy indices (they are summed over), this means that the sum is minus itself, hence zero. So that settles it.

5. I was so fixated on finding another application of the power series expansion approach that I didn't notice there is a very easy way to do this problem. First, I'll describe the way using the useful formula. It says

$$\frac{1}{|\mathbf{r}|} \approx \frac{1}{|\mathbf{r}_0|} - \frac{\mathbf{r}_0 \cdot \boldsymbol{\epsilon}}{|\mathbf{r}_0|^3} + \frac{1}{|\mathbf{r}_0|^5} \left(3(\mathbf{r}_0 \cdot \boldsymbol{\epsilon})^2 - |\mathbf{r}_0|^2 |\boldsymbol{\epsilon}|^2 \right).$$

In this case, \mathbf{r}_0 runs from a point on the Ringworld back to the center, so

$$\mathbf{r}_0 = -R \cos \theta \hat{\mathbf{e}}_x - R \sin \theta \hat{\mathbf{e}}_y,$$

where θ is the usual polar coordinate of the point, and

$$\boldsymbol{\epsilon} = \epsilon \hat{\mathbf{e}}_z.$$

Since $\mathbf{r}_0 \cdot \boldsymbol{\epsilon} = 0$, the formula simplifies considerably:

$$\frac{1}{|\mathbf{r}|} \approx \frac{1}{R} - \frac{\epsilon^2}{R^3}.$$

The potential created by the ring at the position of the sun is

$$\Phi(\epsilon) \approx (-G\lambda) \int_0^{2\pi} \left(\frac{1}{R} - \frac{\epsilon^2}{R^3} \right) R d\theta = \frac{GM}{R} \left(1 + \frac{\epsilon^2}{R^2} \right).$$

Since the curvature is upward, the point $\epsilon = 0$ is stable.

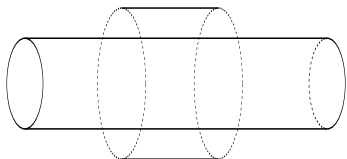
Now the easy way. The distance from the sun to any point on the Ringworld is simply $R^2 + \epsilon^2$, so that

$$\Phi(\epsilon) = (-G\lambda) \int_0^{2\pi} \frac{1}{\sqrt{R^2 + \epsilon^2}} R d\theta = -\frac{2\pi GM}{\sqrt{R^2 + \epsilon^2}}.$$

Clearly, Φ is minimized at $\epsilon = 0$.

One can also see quite directly that the force on a sun shifted in the positive z direction has a negative z component, pulling it back. So there's yet a third way.

6. (a) We use an imaginary ‘‘Gaussian’’ cylinder of radius r and length L coaxial with the material cylinder. By the symmetry of the setup, we can deduce the following: \mathbf{g} is everywhere radial, directed inward, and its magnitude is a function only of the distance from the center of the cylinder. Thus, $\mathbf{g} \cdot \hat{\mathbf{n}}$ is zero on the end caps of our Gaussian cylinder. In any event, the total flux is then $g(r)2\pi aL$. Now we must consider the two cases of inside and outside the cylinder.



For $r > R$, the mass enclosed by the cylinder is $M = \rho\pi R^2 L$, so that

$$2\pi aLg(r) = -4\pi G(\rho\pi R^2 L) \Rightarrow g(r) = -2\pi G\rho R^2 \frac{1}{r}.$$

For $r < R$, the mass enclosed is $M = \rho\pi r^2 L$, so that

$$2\pi aLg(r) = -4\pi G(\rho\pi r^2 L) \Rightarrow g(r) = -2\pi G\rho r.$$

- (b) Unfortunately, it was very easy to completely misunderstand what I was driving at in this part. I should have turned the cylinder into a wire. If you do what I actually said, you may well have reasonably decided that the problem referred to was simply a very gnarly integral.

The issue I was actually interested in was the fact that we cannot take $\Phi(r) \rightarrow 0$ as $r \rightarrow \infty$. This is clear from trying to find the potential difference by integrating \mathbf{g} :

$$\Phi(r_1) - \Phi(r_2) = \int_{r_1}^{r_2} g(r) dr.$$

Since $g \propto -1/r$ outside, if both r_1 and r_2 are bigger than R ,

$$\Phi(r_1) - \Phi(r_2) \propto \ln \frac{r_2}{r_1}.$$

Since the logarithm diverges, it is not possible to choose $\Phi(\infty) = 0$. We must use another reference point.

7. (a) The given vector field is

$$\mathbf{F} = \frac{y\hat{\mathbf{x}} - x\hat{\mathbf{y}}}{r^2} = \frac{1}{r^2} \mathbf{r} \times \hat{\mathbf{z}}.$$

From the last form, it is clear that this vector field goes round and round the origin. In fact, it looks a good deal like Figure 1.3 in the Notes. Thus, $\oint \mathbf{F} \cdot d\ell \neq 0$ for any path around the origin and this is not a conservative force field. This despite the fact that $\nabla \times \mathbf{F}$ is zero away from the origin.

- (b) We are certainly free to choose the orientation of our axes so that $\mathbf{a} = a\hat{\mathbf{e}}_x$. In that case,

$$\mathbf{F} = \mathbf{a}h(\mathbf{a} \cdot \mathbf{r}) = h(ax) \hat{\mathbf{e}}_x,$$

where $\mathbf{r} = x\hat{\mathbf{e}}_x + y\hat{\mathbf{e}}_y$. Since

$$\frac{\partial}{\partial y} h(ax) = 0,$$

this vector field is curl-free everywhere: $\nabla \times \mathbf{F} \equiv 0$. Thus it is conservative.

8. As the projectile ascends, the horizontal (x) component of its momentum does not change since gravity acts vertically. Thus, the x -component of total momentum is computed from the initial velocity as

$$P_x = mv_x(0) = m \frac{v_0}{\sqrt{2}}.$$

The initial kinetic energy of the projectile (just as it's fired) is

$$E_0 = \frac{1}{2}mv_0^2 = 2 \frac{P_x^2}{2m}.$$

Since there is no y -component at the top of the trajectory, the total kinetic energy immediately before the explosion is

$$T = \frac{P_x^2}{2m} = \frac{E_0}{2}.$$

By dealing with immediately before and after, we are able to ignore the gravitational forces on the projectile and the fragments. The explosion provides an additional $E_0/2$ of kinetic energy, so that immediately after, the kinetic energy is

$$T = \frac{3E_0}{2} = T_1 + T_2,$$

and the total momentum is

$$\mathbf{P} = P_x \hat{\mathbf{e}}_x = \sqrt{mE_0} \hat{\mathbf{e}}_x = \mathbf{P}_1 + \mathbf{P}_2.$$

So

$$\mathbf{P}_2 = \sqrt{mE_0} \hat{\mathbf{e}}_x + P_1 \hat{\mathbf{e}}_y \Rightarrow P_2^2 = mE_0 + P_1^2, \quad (5)$$

where P_1 is the magnitude of fragment 1's momentum, which is directed in the $-\hat{\mathbf{e}}_y$ direction.

Now, substituting from Eq. (5) into

$$\frac{3}{2}E_0 = \frac{P_1^2}{2m_1} + \frac{P_2^2}{2m_2}, \quad (6)$$

one finds

$$3E_0 = \left(\frac{1}{m_1} + \frac{1}{m_2} \right) P_1^2 + \frac{m}{m_2} E_0.$$

A bit of algebra will yield P_1 :

$$\left(\frac{3m_2 - m}{m_2} \right) E_0 = \frac{m}{m_1 m_2} P_1^2,$$

finally

$$P_1^2 = \left(2 - 3 \frac{m_1}{m} \right) m_1 E_0.$$

Substituting this back into equation (6) for the total energy,

$$P_2^2 = 2m_2 \left(\frac{3}{2}E_0 - \frac{P_1^2}{2m_1} \right) = \left(1 + 3 \frac{m_1}{m} \right) m_2 E_0.$$

Fragment 2 has an initial velocity directed at an angle θ above the horizontal, and since the total y component of momentum is zero,

$$P_2 \sin \theta = P_1.$$

Thus, the direction in which fragment 2 moves initially is given by

$$\sin \theta = \frac{P_1}{P_2} = \left[\frac{2 - 3m_1/m}{1 + 3m_1/m} \left(\frac{m_1}{m_2} \right) \right]^{1/2}.$$

Finally, the maximum value allowed for m_1 can be found from either the condition that $\theta = 0$, or $P_1^2 = 0$. The result is

$$m_{1,\max} = \frac{2}{3}m.$$