Homework 2 Solutions Physics 419

1. (a) According to the summation convention,

$$\delta_{ij}\delta_{jk} = \delta_{i1}\delta_{1k} + \delta_{i2}\delta_{2k} + \delta_{i3}\delta_{3k}.$$

By the definition of the Kronecker delta, if i and k are identical, precisely one of the terms on the right hand side will be one and the others zero. If i and k are different, all three terms are clearly zero. So, the sum is one if $i = k$, zero otherwise, i.e.

$$\delta_{ij}\delta_{jk} = \delta_{ik}.$$

(b) Since $\partial x_j / \partial x_i = \delta_{ij}$,

$$\nabla(\mathbf{x} \cdot \mathbf{x}) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}(x_j x_j) = \hat{\mathbf{e}}_i 2 x_j \delta_{ij} = 2\hat{\mathbf{e}}_i x_i = 2\mathbf{x}.$$

(c)

$$\nabla(|\mathbf{x}|) = \hat{\mathbf{e}}_i \frac{\partial}{\partial x_i}(x_j x_j)^{1/2} = \hat{\mathbf{e}}_i \frac{1}{2}|\mathbf{x}|^{-1} \frac{\partial}{\partial x_i}(x_j x_j) = \hat{\mathbf{e}}_i \frac{1}{2}|\mathbf{x}|^{-1} 2x_i = \frac{\mathbf{x}}{|\mathbf{x}|}.$$

The first equality is the definition of ∇, the second follows from the chain rule, the third from the observation in part (b) and the last is just switching back to vector notation.

(d) This is an application of $\partial x_i / \partial x_j = \delta_{ij}$ and the product rule.

$$\begin{aligned}
\frac{\partial}{\partial x_i}\frac{x_j}{|\mathbf{x}|} &= \frac{\delta_{ij}}{|\mathbf{x}|} + x_j(-1/2)(|\mathbf{x}|^2)^{-3/2}\frac{\partial}{\partial x_i}(x_k x_k) \\
&= \frac{\delta_{ij}}{|\mathbf{x}|} + x_j(-1/2)(|\mathbf{x}|^2)^{-3/2} 2x_i \\
&= \frac{1}{|\mathbf{x}|^3}(\delta_{ij} - x_i x_j)
\end{aligned}$$

(e) We need to evaluate

$$\varepsilon_{ijk}\varepsilon_{kmn} = \varepsilon_{kij}\varepsilon_{kmn} = \varepsilon_{1ij}\varepsilon_{1mn} + \varepsilon_{2ij}\varepsilon_{2mn} + \varepsilon_{3ij}\varepsilon_{3mn} \qquad (1)$$

for all values of i, j, m and n. Consider the subcases:

i. $i = j$ or $m = n$. In this case, no matter what value k takes, at least one of ε_{kij} or ε_{kmn} will be zero.

ii. $i \neq j$ and $m \neq n$, but the sets $\{i, j\}$ and $\{m, n\}$ are not equal. Then, since there are only three values for the indices, there is no value for k which is different from both i and j at the same time that it is different from m and n. So, in no term of the sum (1) can both ε's be nonzero. The result is zero again.

iii. All that remains is $i = m$ and $j = n$ or $i = n$ and $j = m$ (along with $i \neq j$). In either case, there is one value for k which differs from both i and j (hence from m and n), so that precisely one term in the sum is nonzero. If $i = m$ and $j = n$, then the two ε factors in that term are either both $+1$ or both -1, but in any event, the product is $+1$. If $i = n$ and $j = m$, we will have one ε equal to $+1$ and the other to -1, so that the product is -1.

Coming from the other end, we check the same three cases:

i. $i = j$ or $m = n$. Then $\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm} = 0$ since the two terms are identical apart from sign. For instance $n = m = 3$ would result in $\delta_{i3}\delta_{j3} - \delta_{i3}\delta_{j3}$.

ii. If $i \neq j$ and $m \neq n$, but $\{i, j\} \neq \{m, n\}$ both $\delta_{im}\delta_{jn}$ and $\delta_{in}\delta_{jm}$ are zero.

iii. Finally, if $i \neq j$ concurrently with $i = m$ and $j = n$, then $i \neq n$, so that $\delta_{im}\delta_{jn} = 1$, but $\delta_{in}\delta_{jm} = 0$. Similarly, if $i \neq j$, $i = n$, and $j = m$, $\delta_{im}\delta_{jn} = 0$ and $\delta_{in}\delta_{jm} = 1$.

So, for all possible values of i, j, m and n, $\varepsilon_{kij}\varepsilon_{kmn} = \delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}$.

Well that is a happy thing. Now we must use it for something to justify the effort. By using the formula for the cross product involving the ε-symbol,

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\epsilon_{ijk}\hat{\mathbf{e}}_i a_j b_k) \cdot (\epsilon_{lmn}\hat{\mathbf{e}}_l c_m d_n) = \epsilon_{ijk}\epsilon_{imn} a_j b_k c_m d_n,$$

since $\hat{\mathbf{e}}_i \cdot \hat{\mathbf{e}}_j = \delta_{ij}$. Using our dandy new formula, this becomes

$$(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) a_j b_k c_m d_n = a_j b_k c_j d_k - a_j b_k c_k d_j = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

(f) Since the i component of $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is given by

$$[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i = \varepsilon_{ijk} a_j (\mathbf{b} \times \mathbf{c})_k = \varepsilon_{ijk} a_j \varepsilon_{klm} b_l c_m = (\varepsilon_{ijk}\varepsilon_{klm}) a_j b_l c_m,$$

this is another application of the formula from (e). Continuing,

$$\cdots = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_j b_l c_m = a_j b_i c_j - a_j b_j c_i = (\mathbf{a} \cdot \mathbf{c}) b_i - (\mathbf{a} \cdot \mathbf{b}) c_i.$$

(g) This is almost the same as the previous part, since $\nabla = \hat{\mathbf{e}}_i \partial/\partial x_i$ is much like a vector with i-th component $\partial/\partial x^i$.

$$[\mathbf{v} \times (\nabla \times \mathbf{A})]_i = v_j \frac{\partial}{\partial x_i} A_j - v_j \frac{\partial}{\partial x_j} A_i.$$

We can rewrite the second term as $(\mathbf{v} \cdot \nabla)\mathbf{A}$, but the first cannot be recast into $\nabla(\mathbf{v} \cdot \mathbf{A})$ because it becomes ambiguous what the ∇ operator is differentiating. (it acts only on $\mathbf{A}$).

2. If it were not for the no-slip condition, the small cylinder would have one degree of freedom, one connected to motion of its center of mass and the second to rotation about that center of mass. Denote the angle clockwise from straight down to the point of contact between the two cylinders by θ. That's the center of mass degree of freedom.

Let's put a little paint mark at a point on the rim of the small cylinder. The angle between this mark and the point of contact between the cylinders is α. The angle between the paint spot and the straight down direction is β. The senses of these angles is indicated in the figure. Either of these angles can be taken as coordinate for the rotational degree of freedom. It is going to become locked to the center of mass degree of freedom θ in a moment when we reinstate friction. From the picture it is clear that

$$\beta = \alpha - \theta.$$

The center of mass moves around on a circle of radius $A - a$, so its motion implies a kinetic energy

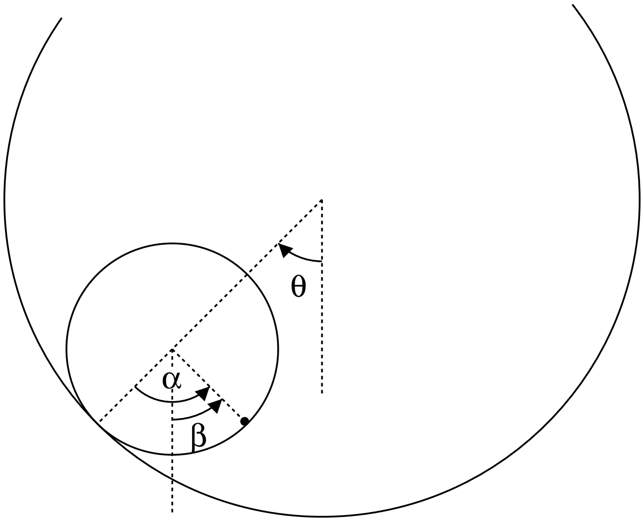
$$T_{\text{CM}} = \frac{m}{2}(A-a)^2\dot{\theta}^2.$$

Rotation of the small cylinder about its center of mass also contributes an amount of kinetic energy

$$T_{\text{rot}} = \frac{m}{2}a^2\dot{\beta}^2.$$

The other ingredient of the Lagrangian is the potential which is simply

$$V = mg(A-a)(1-\cos\theta).$$



Now put the small cylinder at its lowest position and apply the paint at the point of contact, and turn on the friction. Roll the small cylinder up the side of the large one. The point of the point of contact moves a distance $a\alpha$ along the rim of the small cylinder and $A\theta$ along the rim of the large one. By no-slip, these must be equal:

$$a\alpha = A\theta.$$

[This is not strictly true if α goes through a full cycle, but it does not matter since we really only need the time derivative of this, and that is strictly true.] So,

$$\dot{\beta} = \dot{\alpha} - \dot{\theta} = \left(\frac{A}{a} - 1\right)\dot{\theta}.$$

The hard work is over. Gather the pieces together into

$$T = T_{\text{CM}} + T_{\text{rot}} = m(A-a)^2\dot{\theta}^2$$

and then

$$L = m(A-a)^2\dot{\theta}^2 - mg(A-a)(1-\cos\theta).$$

This is just like the Lagrangian for a simple pendulum, except that the coefficients are slightly weird.

$$\ddot{\theta} + \frac{g}{2(A-a)}\sin\theta = 0.$$

3. Let's use upside down spherical coordinates, i.e., spherical coordinates with origin at the support of the pendulum and the z-axis pointing downward. Then θ denotes the angle between straight down and the position vector of the bob and ϕ denotes the angle clockwise from the x-axis to the projection of the position into the x-y plane. Since the square of an differential displacement along a sphere is given by ($dr = 0$)

$$(ds)^2 = (\ell d\theta)^2 + (\ell\sin\theta d\phi)^2$$

the velocity of the bob is

$$|\mathbf{v}|^2 = \left(\frac{ds}{dt}\right)^2 = \ell^2\dot{\theta}^2 + (\ell\sin\theta)^2\dot{\phi}^2.$$

The potential is

$$V = mg\ell(1-\cos\theta)$$

taking the lowest attainable point as the zero. Putting those together, the Lagrangian is

$$L = \frac{m\ell^2}{2}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right)^2 - mg\ell(1-\cos\theta).$$

What we need from this is

$$\begin{aligned}
\frac{\partial L}{\partial\dot{\theta}} &= m\ell^2\dot{\theta} \\
\frac{\partial L}{\partial\dot{\phi}} &= m\ell^2\sin^2\theta\dot{\phi} \\
\frac{\partial L}{\partial\theta} &= m\ell^2\sin\theta\cos\theta\dot{\phi}^2 - mg\ell\sin\theta \\
\frac{\partial L}{\partial\phi} &= 0.
\end{aligned}$$

Collecting the pieces, the Euler-Lagrange equation for θ is

$$\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 + \frac{g}{\ell}\sin\theta = 0.$$

The Euler-Lagrange equation for ϕ is

$$\frac{d}{dt}(\sin^2\theta\dot{\phi}) = 0,$$

which can be written also as

$$2(\cos\theta)\dot{\theta}\dot{\phi} + (\sin\theta)\ddot{\phi} = 0,$$

after dropping a factor of $\sin\theta$ (which is zero only at the points where ϕ is singular anyway). But this is ugly. I like the first form much more.

4. As usual with a pendulum, the angle between straight down and the rod is denoted by θ. The cartesian components of the bob's velocity relative to the support are

$$v_x^{(0)} = -\ell\cos\theta\,\dot{\theta}, \qquad v_y^{(0)} = \ell\sin\theta\,\dot{\theta}.$$

Adding the velocity of the support, $-a\omega\hat{\mathbf{e}}_y\sin\omega t$, to this, the bob's velocity relative to our inertial frame is

$$v_x = -\ell\cos\theta\,\dot{\theta}, \qquad v_y = \ell\sin\theta\,\dot{\theta} - a\omega\sin\omega t.$$

Squaring the two components and adding them together,

$$T = \frac{1}{2}m|\mathbf{v}|^2 = \frac{m\ell^2}{2}\left[\dot{\theta}^2 - 2\frac{a\omega}{\ell}\sin\theta\sin(\omega t)\,\dot{\theta} + \left(\frac{a\omega}{\ell}\right)^2\sin^2(\omega t)\right]$$

The potential energy is $V = mgh$, where h is the height of the bob above some reference point in our inertial reference frame. This involves the position of the support (which is changing) and the height of the bob relative to the support:

$$h = \ell(1-\cos\theta) + a\cos(\omega t).$$

So,

$$\begin{aligned}\frac{\partial L}{\partial\dot{\theta}} &= m\ell^2\left[\dot{\theta} - \frac{a\omega}{\ell}\sin(\omega t)\sin\theta\right]\\ \frac{\partial L}{\partial\theta} = \frac{\partial T}{\partial\theta} - \frac{\partial V}{\partial\theta} &= -m\ell\left(a\omega\cos\theta\sin(\omega t)\dot{\theta} + g\sin\theta\right)\end{aligned}$$

The Lagrange equation is therefore

$$m\ell^2\frac{d}{dt}\left(\dot{\theta} - \frac{a\omega}{\ell}\sin(\omega t)\sin\theta\right) + m\ell\left(a\omega\cos\theta\sin(\omega t)\dot{\theta} + g\sin\theta\right) = 0.$$

Carrying out the differetiation in the first term explicitly, one term cancels with a part of $\partial L/\partial\theta$ to leave

$$\ddot{\theta} = \left(\frac{a\omega^2}{\ell}\cos\omega t - \frac{g}{\ell}\right)\sin\theta.$$

Probably some simple double-checks for errors are in order. Verify that all terms have the same dimension (T^{-2}) and that $a = 0$ (no wiggling of the support) gives back the old familiar pendulum equation. What happens when $g = 0$ is also interesting.

5. The times t_1 and t_2 for the ray to get from A to the interface and then from the interface to B are

$$\begin{aligned} t_1 &= \frac{\sqrt{(L-x)^2+y_1^2}}{c/n_1} \\ t_2 &= \frac{\sqrt{x^2+y_2^2}}{c/n_2}. \end{aligned}$$

The total transit time is therefore

$$T(x) = \frac{n_1}{c}\sqrt{(L-x)^2+y_1^2} + \frac{n_2}{c}\sqrt{x^2+y_1^2}.$$

This depends only upon x, so to minimize, we set dT/dx equal to zero.

$$-\frac{L-x}{\sqrt{(L-x)^2+y_1^2}}n_1 + \frac{x}{\sqrt{x^2+y_2^2}}n_2 = 0.$$

After checking the diagram again, you recognize the coefficients of n_1 and n_2 as being simple trigonometric functions of θ_1 and θ_2, so that

$$-n_1 \sin\theta_1 + n_2 \sin\theta_2 n_2 = 0,$$

which is Snell's Law. (Yay!)

6. (a) The total travel time of the ray is

$$T = \int \frac{d\ell}{v} = \frac{1}{c}\int n(z) d\ell,$$

where

$$(d\ell)^2 = (dx)^2 + (dz)^2 = (dx)^2\sqrt{1+(z')^2}.$$

Inserting that expression into the integral, one obtains the standard variatonal calculus problem: find $z(x)$ to minimize the integral over x of $f(z, z', x) = n(z)\sqrt{1+(z')^2}$. Since

$$\begin{aligned} \frac{\partial f}{\partial z} &= \frac{dn}{dz}\sqrt{1+(z')^2} \\ \frac{\partial f}{\partial z'} &= \frac{n(z)z'}{\sqrt{1+(z')^2}} \end{aligned}$$

the Euler equation is

$$\frac{d}{dx}\left(\frac{n(z)z'}{\sqrt{1+(z')^2}}\right) - \frac{dn}{dz}\sqrt{1+(z')^2} = 0. \qquad (2)$$

Rearranging this into the form given in the problem is a mildly messy exercise in differentiation and algebra:

$$\begin{aligned}\frac{d}{dx}\left(\frac{n(z)z'}{\sqrt{1+(z')^2}}\right) &= \frac{(dn/dz)(z')^2+nz''}{\sqrt{1+(z')^2}}+nz'\frac{-1/2}{[1+(z')^2]^{3/2}}2z'z'' \\ &= \frac{1}{\sqrt{1+(z')^2}}\left[\frac{dn}{dz}(z')^2-\frac{n\,z''}{1+(z')^2}\right]\end{aligned}$$

Putting this back into eq. (2) and dropping a common factor which is never zero,

$$\frac{z''}{1+z'^2}=\frac{dn/dz}{n}.$$

(b) Simply writing down the Euler equation, we have

$$\frac{d}{dx}\frac{\partial f}{\partial y'}-\frac{\partial f}{\partial y}=0. \tag{3}$$

Now, acting on a function of y, y' and x,

$$\frac{d}{dx}=\frac{\partial}{\partial x}+y'\frac{\partial}{\partial y}+y''\frac{\partial}{\partial y'},$$

and if f is actually not explictly dependent on x, the first piece goes away so that

$$\frac{df}{dx}=y'\frac{\partial f}{\partial y}+\frac{dy'}{dx}\frac{\partial f}{\partial y'}.$$

Multiplying the Euler equation (3) through by y', we'll get the first term on the right hand side here, so making that substitution,

$$y'\frac{d}{dx}\frac{\partial f}{\partial y'}-\frac{df}{dx}+\frac{dy'}{dx}\frac{\partial f}{\partial y'}=0$$

But now the first and third terms of this last equation can be combined into a total derivative to finally get

$$\frac{d}{dx}\left[y'\frac{\partial f}{\partial y'}-f\right]=0.$$

(c) Coming back to the case at hand, we just substitute $n(z)\sqrt{1+(z')^2}$ for f and can actually read the required derivative right out of part (a):

$$z'\frac{n(z)z'}{\sqrt{1+(z')^2}}-n(z)\sqrt{1+(z')^2}=\frac{-n}{\sqrt{1+(z')^2}}$$

is independent of x. When the ray reaches the top of its path, $z'=0$, so

$$n(z_{\max})=\frac{n(0)}{\sqrt{1+z'(0)^2}}.$$

Putting in the form $n(z)=n(0)e^{-z/\lambda}$ and taking a logarithm,

$$z_{\max}=\frac{\lambda}{2}\ln[1+z'(0)^2].$$

(d) In the mechanical context, the independent variable x becomes time t, the function y becomes a configuration coordinate q^i and f becomes the Lagrangian $T - U$. Thus, the conserved quantity is

$$H = \sum_i \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L. \tag{4}$$

(H is completely standard notation for this thing, which is called the *Hamiltonian*) In cartesian coordinates, with position $\mathbf{x}_\alpha$ for particle α,

$$L = \sum_\alpha \frac{m_\alpha}{2} |\dot{\mathbf{x}}_\alpha|^2 - U,$$

so that

$$\sum_\alpha \dot{\mathbf{x}}_\alpha \cdot \frac{\partial L}{\partial \dot{\mathbf{x}}_\alpha} = \sum_\alpha m_\alpha |\dot{\mathbf{x}}_\alpha|^2 = 2T.$$

The conserved quantity of eqn. (4) is then

$$2T - (T - U) = T + U = E,$$

the total energy.

7. (a) Really, it's best to use the result of part 1(g). If you insist on doing it the hard way...

$$[\mathbf{v}\times(\nabla\times\mathbf{A})]_x = v_y(\nabla\times\mathbf{A})_z - v_z(\nabla\times\mathbf{A})_y = v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right).$$

Since y and z play perfectly symmetrical rôles here (i.e. we could switch all y's for z's and vice versa without changing the result), it is clear that the y component is to be had by changing x to y and putting x and z in for y and z. All together

$$\begin{aligned}
\mathbf{v}\times(\nabla\times\mathbf{A}) &= \left[v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]\hat{\mathbf{e}}_x \\
&+ \left[v_x\left(\frac{\partial A_x}{\partial y} - \frac{\partial A_y}{\partial x}\right) - v_z\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\right]\hat{\mathbf{e}}_y \\
&+ \left[v_x\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) - v_y\left(\frac{\partial A_y}{\partial z} - \frac{\partial A_z}{\partial y}\right)\right]\hat{\mathbf{e}}_z.
\end{aligned}$$

Pretty awful, isn't it? In the index method, we just write

$$[\mathbf{v}\times(\nabla\times\mathbf{A})]_i = v_j \frac{\partial}{\partial x_i} A_j - v_j \frac{\partial}{\partial x_j} A_i.$$

(b) The derivative $d\mathbf{A}dt$ is the time derivative along the particle's trajectory. The field $\mathbf{A}$ is defined all over the place, but the only values that matter are those at the position of the particle at the time that it's there. So it is

$$\frac{d}{dt}\mathbf{A}(\mathbf{x}(t), t)$$

and by the chain rule, this is

$$\frac{d}{dt}\mathbf{A}(\mathbf{x}(t),t) = \frac{\partial \mathbf{A}}{\partial t} + \sum_i \frac{\partial \mathbf{A}}{\partial x_i}\frac{dx_i}{dt}.$$

The first term is the intrinsic time variation of the vector potential and the second term is due to the motion of the particle. It is much like watching the reading on a thermometer outside an airplane window. What you see will reflect changes in the temperature field, but even if the temperature is constant everywhere, the reading may change because you're travelling from (say) a warm region into a cold one.

Notice that this can also be written as

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A}.$$

(c)

$$L = \frac{m}{2}|\mathbf{v}|^2 + e(\mathbf{v}\cdot\mathbf{A} - \phi)$$

$$\begin{aligned} \frac{\partial L}{\partial \dot{x}_i} &= m\dot{x}_i + eA_i \\ \frac{\partial L}{\partial x_i} &= e\left(v_j\frac{\partial A_j}{\partial x_i} - \frac{\partial \phi}{\partial x_i}\right) \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} &= m\ddot{x}_i + e\frac{dA_i}{dt} \\ &= m\ddot{x}_i + e\frac{\partial A_i}{\partial t} + e\frac{\partial A_i}{\partial x_j}\dot{x}_j. \end{aligned}$$

(We're back to the summation convention here)

Putting these things together into the Euler-Lagrange equation, and doing a bit of rearrangement along with replacing $\dot{x}_i$ by v_i,

$$m\ddot{x}_i + e\left\{\frac{\partial A_i}{\partial t} + \frac{\partial \phi}{\partial x_i} + v_j\left(\frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}\right)\right\} = 0. \tag{5}$$

From the introduction to the problem, you recognize the first two terms inside the braces are $-E_i$ and from part (a), the last two as the i component of $-\mathbf{v}\times(\nabla\times\mathbf{A}) = -\mathbf{v}\times\mathbf{B}$. This is precisely the Lorentz force, as desired. Yay.

(d) Under the gauge transformation,

$$\begin{aligned} \mathbf{A} &\to \mathbf{A} + \nabla\chi \\ \phi &\to \phi - \frac{\partial \chi}{\partial t}, \end{aligned}$$

the electric and magnetic fields transform as

$$\begin{aligned}\mathbf{E} &\rightarrow \mathbf{E} - \frac{\partial}{\partial t}(\nabla\chi) - \nabla\left(-\frac{\partial\chi}{\partial t}\right) = \mathbf{E}.\\ \mathbf{B} &\rightarrow \mathbf{B} + \nabla\times(\nabla\chi) = \mathbf{B},\end{aligned} \tag{6}$$

the first being obvious and the latter following from

$$[\nabla\times(\nabla\chi)]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}\chi = 0$$

because $\varepsilon_{ijk} = -\varepsilon_{jik}$. So the electric and magnetic fields do not change under a gauge transformation (one calls them *gauge invariant*).

The Lagrangian is not expressed in terms of **E** and **B** however, but in terms of **A** and ϕ, so it changes by

$$\Delta L = e\left(\mathbf{v}\cdot\nabla\chi + \frac{\partial\chi}{\partial t}\right).$$

You immediately recognize again the kind of derivative which showed up in part (b). (This is sometimes called a convective derivative, especially in the context of fluid dynamics). So,

$$\Delta L = \frac{d\chi}{dt}.$$

The change in the action of a trajectory $\mathbf{x}(t)$ from $\mathbf{x}_0$ at time t_0 to $\mathbf{x}_1$ at time t_1 is then

$$\Delta S = \int_{t_0}^{t_1} \Delta L\, dt = \chi(\mathbf{x}_1, t_1) - \chi(\mathbf{x}_0, t_0).$$

So the action, too is altered, but the change is independent of the trajectory so that the difference between the actions of one trajectory and another is not altered and therefore changing χ does not alter wich trajectory makes S stationary and Hamilton's Principle is in that sense also gauge invariant. This is of course a good thing since the gauge transformation is not a physical operation but merely a mathematical one not affecting the observable fields **E** and **B**, so that it would be a disaster if it had any influence on the motion of the particle.

Through exactly the same reasoning, you can see that quite generally the Lagrangian may be altered by the addition of a total time derivative without changing any of the implications of that Lagrangian. So the Lagrangian is actually arbitrary to that extent. This is somewhat like the arbitrary constant in a potential. (which carries over to the Lagrangian as well, note)