1. (a) Nothing to do here.

(b) The combination has three translational degrees of freedom which we might identify with the position of particle 1 (or maybe with the center of mass). It also has two rotational degrees of freedom. Once the position of particle 1 is specified, particle 2 can be anywhere on the surface of a sphere with radius d. Thus, to specify a configuration, we would need a position plus a point on the sphere. This gives us five degrees of freedom. We can arrive at this conclusion also by a negative argument, i.e., start with the six degrees of freedom of two free particles and eliminate one for the constraint. Since the translational and rotational parts of the configuration are completely independent, the configuration space is $\mathcal{Q} = \mathbb{R}^3 \times S^2$.

(c) The cardinal requirement on the constraint function is that it take some (any) constant value on the allowed configurations. The original specification was that $g(\mathbf{r}_1, \mathbf{r}_2) \equiv |\mathbf{r}_1 - \mathbf{r}_2| = d$. Clearly, however, an equivalent condition is $|\mathbf{r}_1 - \mathbf{r}_2|^2 = d^2$. Pretty much any function of $|\mathbf{r}_1 - \mathbf{r}_2|$ will do. (That's not quite true, but we'll ignore that for now)

(d) The kinetic energy is simply the sum of the kinetic energies of the two particles:

$$T = \frac{1}{2} m_1 |\dot{\mathbf{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\mathbf{r}}_2|^2.$$

As a result,

$$\tilde{L} = \frac{1}{2} m_1 |\dot{\mathbf{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\mathbf{r}}_2|^2 - U(\mathbf{r}_1, \mathbf{r}_2) + \lambda |\mathbf{r}_1 - \mathbf{r}_2|.$$

(e) The ingredients needed for the Lagrange equations of motion are

$$\frac{\partial \tilde{L}}{\partial \dot{\mathbf{r}}_1} = \mathbf{p}_1, \qquad \frac{\partial \tilde{L}}{\partial \dot{\mathbf{r}}_2} = \mathbf{p}_2,$$
$$\frac{\partial \tilde{L}}{\partial \mathbf{r}_1} = -\frac{\partial U}{\partial \mathbf{r}_1} + \lambda \frac{\mathbf{r}}{r}, \qquad \frac{\partial \tilde{L}}{\partial \mathbf{r}_2} = -\frac{\partial U}{\partial \mathbf{r}_2} - \lambda \frac{\mathbf{r}}{r},$$

This uses the abbreviations $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, $r = |\mathbf{r}_1 - \mathbf{r}_2|$, and

$$\frac{\partial r}{\partial \mathbf{r}_1} = \frac{\mathbf{r}}{r} \quad \frac{\partial r}{\partial \mathbf{r}_2} = -\frac{\mathbf{r}}{r},$$

as follows from the chain rule. I also write

$$\frac{\partial}{\partial \mathbf{r}_1} = \sum_i \hat{\mathbf{e}}_i \frac{\partial}{\partial r_{1,i}} = \hat{\mathbf{e}}_x \frac{\partial}{\partial x_1} + \hat{\mathbf{e}}_y \frac{\partial}{\partial y_1} + \hat{\mathbf{e}}_z \frac{\partial}{\partial z_1}$$

The Lagrangian equations are therefore

$$\begin{aligned} \dot{\mathbf{p}}_1 &= -\frac{\partial U}{\partial \mathbf{r}_1} + \lambda \frac{\mathbf{r}}{r}, \\ \dot{\mathbf{p}}_2 &= -\frac{\partial U}{\partial \mathbf{r}_2} - \lambda \frac{\mathbf{r}}{r}. \end{aligned} \tag{1}$$

The first terms on the right hand side are the forces due to the potential U. Newton's 2nd Law assures us that the terms involving λ must therefore be the constraint force since there is no other force operating. The constraint forces are

$$\mathbf{f}_1 = \lambda\hat{\mathbf{r}}, \quad \mathbf{f}_2 = -\lambda\hat{\mathbf{r}}.$$

Since these are equal and opposite (and directed along the separation vector to boot), they are consistent with the strong form of Newton's 3rd Law.

(f) Using

$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2,$$

substitute into

$$M\mathbf{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2,$$

first for $\mathbf{r}_1$, then for $\mathbf{r}_2$ to get

$$\begin{aligned} M\mathbf{R} &= m_1\mathbf{r}_1 + (m_1 + m_2)\mathbf{r}, \\ M\mathbf{R} &= (m_1 + m_2)\mathbf{r} - m_2\mathbf{r}_2. \end{aligned} \tag{2}$$

Isolating $\mathbf{r}$ on one side, and recognizing $\mathbf{r}'_1$ and $\mathbf{r}'_2$,

$$\begin{aligned} \mathbf{r}'_1 &= \frac{m_2}{M}\mathbf{r}, \\ \mathbf{r}'_2 &= -\frac{m_1}{M}\mathbf{r}. \end{aligned}$$

Now we get $\mathbf{r}_1$ and $\mathbf{r}_2$ just by adding $\mathbf{R}$ back in:

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{R} + \frac{m_2}{M}\mathbf{r}, \\ \mathbf{r}_2 &= \mathbf{R} - \frac{m_1}{M}\mathbf{r}. \end{aligned} \tag{3}$$

It is not really surprising that $\mathbf{r}'_1$ and $\mathbf{r}'_2$ are linearly dependent because we started with six degrees of freedom, and three are taken up by $\mathbf{R}$. So there is really only one vector's worth after that. Another way to look at it is to put the origin of coordinates at the center of mass. That's exactly what we do when using the primed vectors. Since the center of mass vector is then zero, the two position vectors $\mathbf{r}'_1$ and $\mathbf{r}'_2$ must be back-to-back.

(g) Applying dots to $\mathbf{r}_1 = \mathbf{R} + \mathbf{r}'_1$, squaring and multiplying by m_1, and similarly for $\mathbf{r}_2$, we get

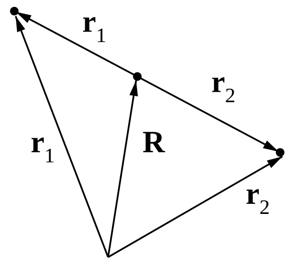
$$2T = m_1|\dot{\mathbf{R}}|^2 + 2m_1\dot{\mathbf{r}}'_1 \cdot \dot{\mathbf{R}} + m_1|\dot{\mathbf{r}}'_1|^2 + m_2|\dot{\mathbf{R}}|^2 + 2m_2\dot{\mathbf{r}}'_2 \cdot \dot{\mathbf{R}} + m_2|\dot{\mathbf{r}}'_2|^2$$

However, since

$$m_1\dot{\mathbf{r}}'_1 + m_2\dot{\mathbf{r}}'_2 = 0,$$

two of these terms combine to give zero, leaving

$$T = \frac{M}{2}|\dot{\mathbf{R}}|^2 + m_1|\dot{\mathbf{r}}'_1|^2 + m_2|\dot{\mathbf{r}}'_2|^2.$$



The general result which this exemplifies is that the kinetic energy of a system of particles is equal to the kinetic energy it would have if all the mass were situated at the center of mass and moving with it, plus kinetic energies computed from the particles' positions relative to the center of mass. As an equation,

$$T = \frac{M}{2}|\dot{\mathbf{R}}|^2 + \sum_\alpha m_\alpha |\dot{\mathbf{r}}'_\alpha|^2.$$

To express this in terms of $\mathbf{R}$ and $\mathbf{r}$, apply dots all around to equations (3) and square:

$$\begin{aligned} |\dot{\mathbf{r}}_1|^2 &= |\dot{\mathbf{R}}|^2 + 2\frac{m_2}{M}\dot{\mathbf{r}}\cdot\dot{\mathbf{R}} + \frac{m_2^2}{M^2}|\dot{\mathbf{r}}|^2 \\ |\dot{\mathbf{r}}_2|^2 &= |\dot{\mathbf{R}}|^2 - 2\frac{m_1}{M}\dot{\mathbf{r}}\cdot\dot{\mathbf{R}} + \frac{m_1^2}{M^2}|\dot{\mathbf{r}}|^2. \end{aligned}$$

Multiply the first equation by $m_1/2$ and the second by $m_2/2$ and add them together. There is a cancellation coming from the cross terms:

$$T = \frac{1}{2}M|\dot{\mathbf{R}}|^2 + \frac{\mu}{2}|\dot{\mathbf{r}}|^2.$$

The second piece was obtained by using

$$\frac{m_1^2 m_2 + m_2^2 m_1}{M^2} = \frac{(m_1+m_2)m_1 m_2}{M^2} = \frac{m_1 m_2}{M} = \mu.$$

(h) Combining the kinetic energy from the last part with the potential rewritten in terms of $\mathbf{R}$ and $\mathbf{r}$,

$$\tilde{L} = \frac{M}{2}|\dot{\mathbf{R}}|^2 + \frac{\mu}{2}|\dot{\mathbf{r}}|^2 - U(\mathbf{R} + m_2\mathbf{r}/M, \mathbf{R} - m_1\mathbf{r}/M) + \lambda r.$$

The derivatives with respect to velocities are easy:

$$\begin{aligned} \frac{\partial \tilde{L}}{\partial \dot{\mathbf{R}}} &= M\dot{\mathbf{R}} = \mathbf{P}, \\ \frac{\partial \tilde{L}}{\partial \dot{\mathbf{r}}} &= \mu\dot{\mathbf{r}} = \mathbf{p}. \end{aligned}$$

And those with respect to positions are not much worse,

$$\begin{aligned}\frac{\partial \tilde{L}}{\partial \mathbf{R}} &= -\frac{\partial \tilde{U}}{\partial \mathbf{r}_1} - \frac{\partial \tilde{U}}{\partial \mathbf{r}_2} \\ \frac{\partial \tilde{L}}{\partial \mathbf{r}} &= -\frac{m_2}{M}\frac{\partial \tilde{U}}{\partial \mathbf{r}_1} + \frac{m_1}{M}\frac{\partial \tilde{U}}{\partial \mathbf{r}_2} + \lambda \hat{\mathbf{r}}.\end{aligned}$$

This yields the equations of motion

$$\begin{aligned}\dot{\mathbf{P}} &= -\frac{\partial \tilde{U}}{\partial \mathbf{r}_1} - \frac{\partial \tilde{U}}{\partial \mathbf{r}_2} \\ \dot{\mathbf{p}} &= -\frac{m_2}{M}\frac{\partial \tilde{U}}{\partial \mathbf{r}_1} + \frac{m_1}{M}\frac{\partial \tilde{U}}{\partial \mathbf{r}_2} + \lambda \hat{\mathbf{r}}.\end{aligned}$$

(i) If we set $U \equiv 0$, the equations of the previous part become

$$\begin{aligned}\dot{\mathbf{P}} &= 0, \\ \mu \ddot{\mathbf{r}} &= \lambda \hat{\mathbf{r}}.\end{aligned}$$

Solving this for $\ddot{\mathbf{r}}$ and substituting into

$$\frac{1}{2}\frac{d^2}{dt^2}|\mathbf{r}|^2 = |\dot{\mathbf{r}}|^2 + \mathbf{r}\cdot\ddot{\mathbf{r}} = 0,$$

the result for λ is

$$\lambda = \mu \frac{|\dot{\mathbf{r}}|^2}{r}.$$

(j) With $m_1 = m_2 = m$, $\mu = m/2$, the result of the part (i) becomes $\lambda = m|\dot{\mathbf{r}}|^2/2r$. In the center of mass frame, we have the pair of particles at the end of the rod rotating around its center. You know the centripetal force on each is $F_{\text{cent}} = mv^2/r/2$, since each is $r/2$ from the center. Since also $v = |\dot{\mathbf{r}}|/2$, λ turns out to equal the centripetal force, $F_{\text{cent}} = \lambda$.

2. (a) Since The kinetic energy is found by

$$T = \frac{m}{2}(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2)$$

(b) The potential actually depends upon θ and z only in the combination $\theta - zd/2\pi$. Thus, it is constant along curves

$$\theta - 2\pi z/d = \theta_0, \quad \rho = \rho_0,$$

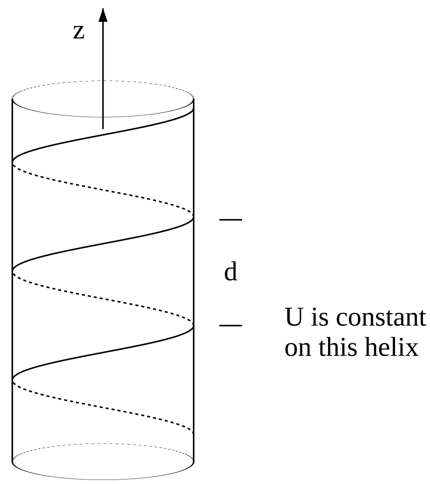
where ρ_0 and θ_0 locate the intersection of the curve with the $z = 0$ plane. To stay on such a curve, it must be that

$$dz = \frac{d}{2\pi}d\theta \Rightarrow \quad \frac{dz}{ds} = \frac{d}{2\pi}\frac{d\theta}{ds}. = \frac{d}{2\pi},$$

and also that

$$\frac{d\rho}{ds} = 0.$$

Since ρ doesn't change as s is cranked up, but z changes proportionally to θ, the curve traced out by a point with increasing s is a helix running along the z direction.



(c) This motion is a combination of a rotation and a translation. Since that does not alter distances, neither does it alter velocities. If you don't trust that, you can just put dots on the transformation equations:

$$\frac{d\dot{\rho}}{ds} = \frac{d\dot{\theta}}{ds} = \frac{d\dot{z}}{ds} = 0.$$

(d) According to Noether's Theorem, the conserved quantity is

$$\text{conserved quantity} = m\left(\dot{\rho}\frac{d\rho}{ds} + \rho^2\dot{\theta}\frac{d\theta}{ds} + \dot{z}\frac{dz}{ds}\right).$$

Plugging in what we got for the derivatives with respect to s, this comes out to be

$$\text{conserved quantity} = m\left(\rho^2\dot{\theta} + \frac{2\pi}{d}\dot{z}\right).$$

(e) Now we can recognize the first term of this as angular momentum about the z axis and the second as the z component of linear momentum. So our conserved quantity is

$$L_z + \frac{2\pi}{d}p_z.$$

(f) What the conservation law in the previous part says is this. A decrease in L_z must be compensated by an increase in p_z. Suppose it were just the angular momentum L_z which was conserved. Then, if the particle moved inward toward the z-axis, it would be obliged to increase its azimuthal speed $\rho\dot{\theta}$ in a compensating manner. In the situation we have here, there is another change which can compensate: an increase in its velocity $\dot{z}$ along the z-direction. Conversely, it's velocity along z could decrease if it picked up azimuthal speed or moved out away from the z-axis. This does not mean that a randomly chosen motion satisfying these constraints can actually occur. The actual motion is determined by initial conditions and details of U which haven't been specified. But it must come from the class of motions which do conserve $L_z + \frac{2\pi}{d}p_z$.