Homework 4 Solutions   Physics 419

1. **imaginary exponents**. Prove the double angle formulas

$$\sin(2\theta) = 2\sin\theta\,\cos\theta, \quad \cos(2\theta) = \cos^2\theta - \sin^2\theta.$$

Straight from the exponential expression for sine,

$$\begin{aligned}\sin(2\theta) &= \frac{e^{i2\theta} - e^{-i2\theta}}{2i}\\ &= 2\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)\left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)\\ &= 2\sin\theta\,\cos\theta.\end{aligned}$$

Similarly,

$$\begin{aligned}\cos(2\theta) &= \frac{e^{i2\theta} + e^{-i2\theta}}{2}\\ &= \frac{1}{4}\left[e^{i2\theta} + 2 + e^{-i2\theta} + \left(e^{i2\theta} - 2 + e^{-i2\theta}\right)\right]\\ &= \cos^2\theta - \sin^2\theta.\end{aligned}$$

2. **Some observation**. I seem to be seeing pendulums everywhere I go these days. For example, the pedals on my bicycle (I have toe clips which make them asymmetric). They have fairly low damping as evidenced by the fact that the swinging takes a long time to die out. At large angles, the dynamics is nonlinear. If I make them spin completely around, that's obvious. But otherwise I know that because I know about the ordinary pendulum.

I saw some tree branches being forced by the wind. That's an oscillator too, and pretty heavily damped as the oscillation dies out after a couple cycles or so. The branch definitely has more than one degree of freedom, and depending on the direction from which the wind blows, could oscillate in different modes.

3. **A toppling pencil**.

(a) To compute the energy of a rod rotating around a perpendicular axis passing through its center, we measure ℓ from the axis, so that the part of the rod at ℓ is moving at a speed $\omega\ell$, and a small length $d\ell$ at that point has kinetic energy $(M/2L)d\ell(\omega\ell)^2$. Adding this up over the whole rod,

$$T_{\text{center}} = \int_{-L/2}^{L/2} \frac{M}{2L}(\omega l)^2 dl = \frac{M}{2L}\omega^2\frac{2}{3}\frac{L^3}{8} = \frac{ML^2}{24}\omega^2.$$

If instead, we compute the energy for rotation about the end,

$$T_{\text{end}} = \int_0^L \frac{M}{2L}(\omega l)^2 dl = \frac{M}{2L}\omega^2\frac{L^3}{3} = \frac{ML^2}{6}\omega^2.$$

(b) If we have computed the rotational energy by using the axis at the center of the pencil, we must still add in the kinetic energy due to the center of mass motion. This is $MV^2/2 = M\omega^2/2(L/2)^2 ML^2\omega^2/8$. Using the angle θ between the pencil and the vertical as our coordinate, $\omega = \dot{\theta}$, so that the total kinetic energy is

$$T = \frac{M}{2}\left(\frac{L\dot{\theta}}{2}\right)^2 + \frac{ML^2}{24}\dot{\theta}^2 = \frac{ML^2}{6}\dot{\theta}^2.$$

Using the rotational energy about the end, there is nothing to add. Fortunately the results obtained by the two methods agree. At any rate, the potential energy is simply

$$U = Mg\frac{L}{2}\cos\theta.$$

Linearizing the potential around the unstable fixed point at $\theta = 0$,

$$U \approx Mg\frac{L}{2}\left(1 - \theta^2/2\right),$$

and dropping a constant,

$$L_{\text{linearized}} = \frac{ML^2}{6}\dot{\theta}^2 + \frac{MgL}{4}\theta^2.$$

The EOM resulting from this is

$$\frac{ML^2}{3}\ddot{\theta} - \frac{MgL}{2}\theta = 0,$$

or

$$\ddot{\theta} - \lambda^2\theta = 0,$$

with $\lambda^2 = 3g/2L$.

$$\theta(t) = ae^{\lambda t} + be^{-\lambda t}$$

The initial condition $\dot{\theta}(0) = 0$ leads to $a = b$, so

$$\theta(t) = \theta(0)\cosh\lambda t.$$

For large times this is approximately

$$\theta(t) \approx (\theta(0)/2)e^{\lambda t}.$$

Solved for t, we find

$$t = \frac{1}{\lambda}\ln\left(2\frac{\theta(t)}{\theta(0)}\right)$$

I measured a pencil to be $L = 19$ cm long, so $\lambda \approx \sqrt{75}\ \text{s}^{-1}$. As a result,

$$t \approx (0.12\,\text{s})\ln(10800) = 1.1\,\text{s}.$$

Notice that this is much more sensitive to g and L than the angle we choose to designate as fallen.

(c) When I try to balance a pencil on its tip, it seems to take a few tenths of a second to fall. To account for that purely from the dynamics we have here, I would need to increase $\theta(0)$ by a factor of 20 to 1/3 degree. It could well be that I can't really balance a pencil to much better than a degree; I'm not sure.

4. **wheels revisited.** The equation of motion for the wheels within wheels system is

$$\ddot{\theta} + \frac{g}{2(A-a)} \sin\theta = 0.$$

(a) Linearization about $\theta = 0$ is easy. We just expand the sine:

$$\ddot{\theta} + \frac{g}{2(A-a)}\theta = 0.$$

So, the frequency of small oscillations is

$$\omega = \sqrt{\frac{g}{2(A-a)}}.$$

(b) When the cylinder falls through a certain angle, its weight does the same amount of work as if it were a simple bob of the same mass. However, this work must go into increasing the rotational energy as well as the translational kinetic energy. As a result, the translational energy does not increase by as much as it would if there were no rotation.

5. **Bead on a bent wire.**

(a) To compute the frequency of *small* oscillations, we want to linearize the Lagrangian around the fixed point $\dot{x} = x = 0$ on the grounds that that is adequate if the amplitude of the motion is small. The line element along the wire is given by

$$ds^2 = dz^2 + dx^2 = \left[1 + \left(\frac{dz}{dx}\right)^2\right] dx^2.$$

Thus, the kinetic energy of the bead is

$$T = \frac{m}{2}\left(\frac{ds}{dt}\right)^2 = \frac{m}{2}\left[1 + \left(\frac{dz}{dx}\right)^2\right]\dot{x}^2.$$

Now we want to expand this to second order in x and $\dot{x}$, but there are already two powers of $\dot{x}$, so we simply evaluate the prefactor $[1 + (dz/dx)^2]$ at the fixed point $x = 0$. Since the wire is horizontal there, $dz/dx|_{x=0} = 0$ and all that survives is the 1:

$$T \approx \frac{m}{2}\dot{x}^2.$$

Carrying out the same procedure with the potential,

$$U = mgz = mg\left(z(0) + z'(0)x + z''(0)\,x^2/2 + \cdots\right) \approx mg\left.\frac{d^2z}{dx^2}\right|_0 \frac{x^2}{2}.$$

Putting these ingredients together,

$$L_{\text{linearized}} = \frac{m}{2}\dot{x}^2 - mg\left.\frac{d^2z}{dx^2}\right|_0 \frac{x^2}{2},$$

leading to an equation of motion

$$\ddot{x} + g\left.\frac{d^2z}{dx^2}\right|_0 x = 0.$$

From this we can immediately identify the frequency. It's

$$\omega_0 = \sqrt{gz''(0)}.$$

(b) If we use arc length s as our coordinate instead, the kinetic energy is simply

$$T = \frac{m}{2}\dot{s}^2$$

as already noticed. But the potential energy is a mess. Fortunately, expanding that in a Taylor series in s will save the day. Since we can take x to be a function of s or vice-versa,

$$\frac{dx}{ds} = \left(\frac{ds}{dx}\right)^{-1} = \left[1 + \left(\frac{dz}{dx}\right)^2\right]^{-1}.$$

Then, the expansion of z in powers of s is

$$z(x(s)) = z(0) + z'(0)\frac{ds}{dx}s + \left[z''(0)\left(\frac{ds}{dx}\right)^2 + z'(0)\frac{d^2s}{dx^2}\right]\frac{s^2}{2} + \cdots$$

We did not compute the second derivative of s with respect to x, but it doesn't matter because the thing multiplying it is zero. The quadratic approximation to the potential is

$$U \approx mgz''(0)\frac{s^2}{2}.$$

Clearly enough, this is going to lead to the same frequency, since the potential and kinetic energies look exactly the same as before, simply with x replaced by s. And that's a good thing.