Homework 4 Solutions Physics 419

1. Practice with vector derivatives.
Three masses m_1, m_2 and m_3 are subject to a potential

$$U(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = k[(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{r}_3 - \mathbf{r}_1)]^2.$$

(a) Describe the set of configurations on which U achieves its minimum.
The potential is minimized when $(\mathbf{r}_2 - \mathbf{r}_1) \cdot (\mathbf{r}_3 - \mathbf{r}_1)$ is zero. This means that $\mathbf{r}_2 - \mathbf{r}_1$ is perpendicular to $\mathbf{r}_3 - \mathbf{r}_1$. Since it doesn't matter how long these separation vectors, but only on the angle between them, this is not a very realistic potential for much of anything.

(b) Compute the forces on all three particles.
Since U actually depends on only $\mathbf{r}_{12}$ and $\mathbf{r}_{13}$,

$$\frac{\partial U}{\partial \mathbf{r}_1} = \frac{\partial U}{\partial \mathbf{r}_{12}} + \frac{\partial U}{\partial \mathbf{r}_{13}},$$

$$\frac{\partial U}{\partial \mathbf{r}_2} = -\frac{\partial U}{\partial \mathbf{r}_{12}}$$

and

$$\frac{\partial U}{\partial \mathbf{r}_3} = -\frac{\partial U}{\partial \mathbf{r}_{13}}.$$

In these expressions, it is understood that the partial derivatives with respect to separation vectors are computed with the other separation vectors held fixed. Those partial derivatives are

$$\begin{aligned} \frac{\partial U}{\partial \mathbf{r}_{12}} &= 2k(\mathbf{r}_{12} \cdot \mathbf{r}_{13})\mathbf{r}_{13}, \\ \frac{\partial U}{\partial \mathbf{r}_{13}} &= 2k(\mathbf{r}_{12} \cdot \mathbf{r}_{13})\mathbf{r}_{12}. \end{aligned}$$

Putting them together,

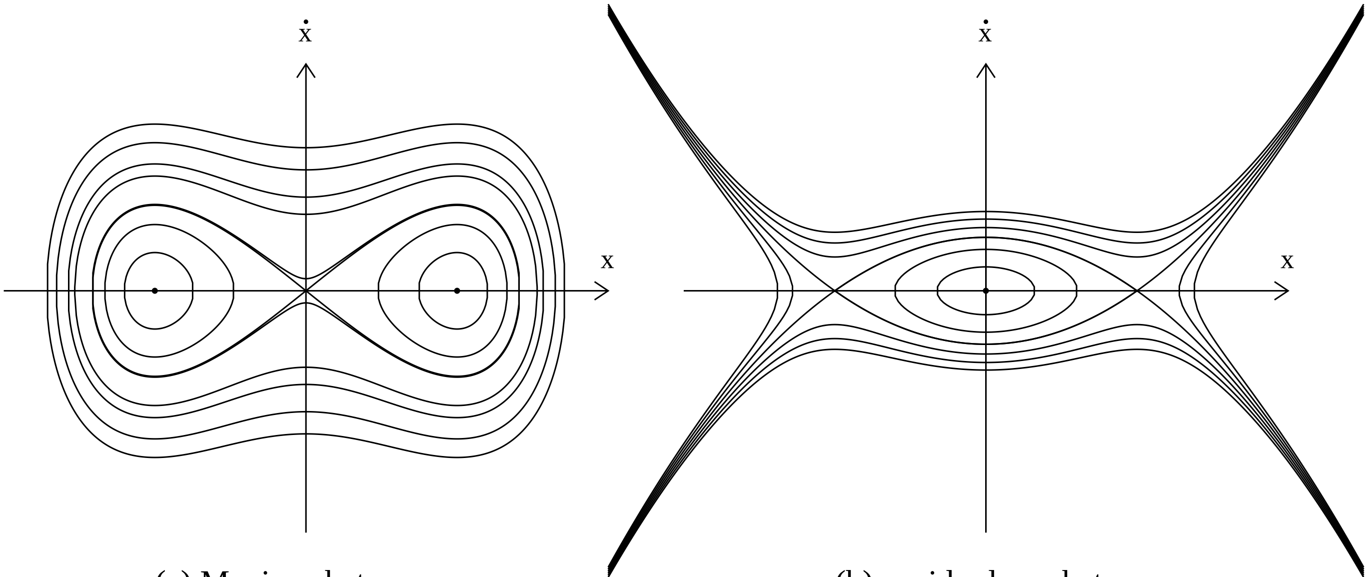
$$\begin{aligned} \mathbf{F}_1 &= -2k(\mathbf{r}_{12} \cdot \mathbf{r}_{13})(\mathbf{r}_{13} + \mathbf{r}_{12}) \\ \mathbf{F}_2 &= 2k(\mathbf{r}_{12} \cdot \mathbf{r}_{13})\mathbf{r}_{13} \\ \mathbf{F}_2 &= 2k(\mathbf{r}_{12} \cdot \mathbf{r}_{13})\mathbf{r}_{12} \end{aligned}$$

2. Mexican hat potential.
Study the motion of a particle in the potential

$$\frac{U_{\text{hat}}(x)}{U_0} = -\left(\frac{x}{a}\right)^2 + \frac{1}{2}\left(\frac{x}{a}\right)^4,$$

and $U_{\text{ud}}(x) = -U_{\text{hat}}(x)$.

(a) Draw phase portraits. I have exploited the fact that the orbits coincide with the curves of constant energy to draw the phase portraits in the accompanying figure. The Mexican hat is shown in (a), using adimensionalized position, velocity and energy as in part (b). The curves shown correspond to $E = -0.4$, -0.2, 0, 0.2, 0.6, 1.0 and 1.4. The



(a) Mexican hat (b) upside-down hat

portraint for the upside-down hat is in (b), for energies 0.1, 0.3, 0.5, 0.7, 0.9, and 1.1. Notice that the high energy curves are crowded together on the left and right edges of the pictures because the potential is very steep there.

(b) Find the equation of motion and clean it up.
The force on the particle is

$$\mathbf{F} = -\frac{dU_{ud}}{dx} = -\frac{U_0}{a} 2\frac{x}{a}\left(1 - \frac{x^2}{a^2}\right)$$

so that the equation of motion is

$$\frac{ma}{U_0}\frac{d^2x}{dt^2} = -2\frac{x}{a}\left(1 - \frac{x^2}{a^2}\right).$$

Defining $\overline{x} = x/a$, this is

$$\frac{ma^2}{U_0}\frac{d^2\overline{x}}{dt^2} + 2\overline{x}(1 - \overline{x}^2) = 0.$$

In order for this equation to be correct, the dimensions of ma^2/U_0 must be T^2 to make the left-hand side dimensionless. It is easy to check this directly, too, since $[U_0] = ML^2T^{-2}$. So, using $t_0 = \sqrt{ma^2/U_0}$ as a unit of time with $\overline{t} = t/t_0$, the equation of motion becomes

$$\frac{d^2\overline{x}}{d\overline{t}^2} + 2\overline{x}(1 - \overline{x}^2) = 0.$$

(c) Linearize around the unstable equilibrium at $x = -1$.
To study motion in the vicinity of the unstable equilibrium point at $x = -1$, define $y = x + 1$, so that $y = 0$ at the equilibrium. Then

$$2x(1 - x^2) = 2x(1 - x)(1 + x) = 2(1 + y)(2 - y)y = 4y + \mathcal{O}(y^2).$$

The linearized equation of motion for y and its general solution are then

$$\ddot{y} = 4y \Rightarrow y = ae^{2t} + be^{-2t} + c.$$

Now, we require that $y \to 0$ as $t \to -\infty$, so that $b = c = 0$ and $a = y(0)$. Rewriting in terms of x,

$$x(t) = -1 + y(0)e^{2t}.$$

If $x(0) = 0$,

$$x(t) = -1 + e^{2t}.$$

(d) Do the same near $x = 1$.
This is the same game. Defining $y = 1 - x$, $y = 0$ at the right-hand unstable equilibrium at $x = 1$. Similarly to before,

$$2x(1 - x^2) = 2x(1 + x)(1 - x) = 2(1 - y)(2 - y)y = 4y + \mathcal{O}(y^2).$$

This gives us exactly the same equation for y with the same general solution. However, this time we impose $y \to 0$ as $t \to \infty$ and $y(0) = -1$. This means that $a = c = 0$ so that $y \to 0$ and then $y(0) = b$. Since we want $y(0) = 1$,

$$x(t) = 1 - e^{-2t}.$$

(e) Demonstrate that the exact solution which interpolates between the two unstable fixed points is

$$x(t) = \tanh t,$$

where

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^z - e^{-z}}{e^z + e^{-z}}.$$

To check that this really is a solution, we check the derivatives,

$$\begin{aligned} \frac{d}{dt}\tanh t &= 1 - \frac{\sinh^2 t}{\cosh^2 t} = \frac{1}{\cosh^2 t} \\ \frac{d^2}{dt^2}\tanh t &= \frac{d}{dt}\left(\frac{1}{\cosh^2 t}\right) = -2\frac{\sinh t}{\cosh^2 t}. \end{aligned}$$

The relation $\cosh^2 t - \sinh^2 t = 1$ was used here. Now, since

$$\begin{aligned} -2\frac{\sinh t}{\cosh^2 t} &= -2\tanh t\frac{\cosh^2 t - \sinh^2 t}{\cosh^2 t}. \\ &= -2\tanh t(1 - \tanh^2 t). \end{aligned} \tag{1}$$

This checks that $\ddot{x} = -2x(1 - x^2)$.

(f) Compare the exact motion to that in the quadratic approximations to U.
The hyperbolic tangent can be rewritten in these two ways:

$$\tanh t = \frac{e^t - e^{-t}}{e^t + e^{-t}} = \frac{1 - e^{-2t}}{1 + e^{-2t}} = -\frac{1 - e^{2t}}{1 + e^{2t}}.$$

Since $e^{2t} \to 0$ as $t \to -\infty$, in that limit

$$-\frac{1 - e^{2t}}{1 + e^{2t}} = -(1 - e^{2t})(1 - e^{2t} + \cdots) \to -1 + 2e^{-2t}.$$

On the other side, Since $e^{-2t} \to 0$ as $t \to \infty$, so that in that limit

$$\frac{1 - e^{-2t}}{1 + e^{-2t}} = (1 - e^{-2t})(1 - e^{-2t} + \cdots) \to 1 - 2e^{-2t}.$$

The solutions in parts (c) and (d) are for a purely quadratic potential. In fact, the potential is shallower than that. Let us compare Bob moving in the quadratic potential fitted to the left-hand hump with Alice moving in the real potential. With $y = x + 1$, from the results of part (c),

$$\begin{aligned} \ddot{y}_b &= 4y_b, \\ \ddot{y}_a &= 4y_a(1 - y_a)\left(1 - \frac{y_a}{2}\right). \end{aligned}$$

For $0 < y$, the force in the real potential is always smaller than in the quadratic one. If Alice and Bob are at the same position with the same velocity at some very early time $t \ll 0$, Bob will always be experiencing a greater acceleration and Alice will fall behind, so that they will not both reach $x = 0$ at the same time. Alice must have a head start. A completely analogous argument applies for the $t \to \infty$ motion near $x = 1$.
Notice that Bob gets ahead by being dishonest.