Homework 6 Solutions

1. similarity transformations.

A particle moves in a potential

$$U(\mathbf{r}) = \frac{k}{m} r^{\alpha},$$

from which follows the equation of motion

$$\frac{d^2\mathbf{r}}{dt^2} + \alpha k r^{\alpha-1}\hat{\mathbf{e}}_r = 0. \tag{1}$$

(a) Define

$$\mathbf{R} = \lambda\mathbf{r}, \quad \tau = \lambda^z t, \tag{2}$$

and find the value of z which makes equation (1) true if $\mathbf{r}$,t are replaced by $\mathbf{R}$,τ.

Taking the left-hand side of the desired equation and making substitutions according to eqn. (2),

$$\frac{d^2\mathbf{R}}{d\tau^2} + \alpha k|\mathbf{R}|^{\alpha-1}\hat{\mathbf{e}}_r = \frac{d^2\lambda\mathbf{r}}{d(\lambda^z t)^2} + \alpha k\lambda|\mathbf{r}|^{\alpha-1}\hat{\mathbf{e}}_r.$$

This will be a multiple of the left hand side of the original equation of motion (1) if the powers of λ in the two terms match. So, we require $\lambda\lambda^{-2z} = \lambda^{\alpha-1}$ which implies

$$z = 1 - \alpha/2.$$

(b) Apply to a simple harmonic oscillator.

The simple harmonic oscillator potential is $V \propto x^2$, which corresponds to $\alpha = 2$. So the rescaled and rescheduled trajectory will also satisfy the equation of motion if $z = 1 - 2/2 = 0$. This means

$$x'(t) = \lambda x(t)$$

is also a solution. x returns to its original value after one period τ, hence so does x'. Thus, the period of the harmonic oscillator is independent of its amplitude.

(c) Interpret the result of part (a) in words.

My description of this is rather verbose, but I think the question a little bit slippery, so it's best to be sure. The method employed in part (a) was rather mechanical. If you think a little more about it, it is quite easy to become confused, especially since the notation there is possibly a little ambiguous. The starting assumption is that $t \mapsto \mathbf{r}_0(t)$ is a solution of the equation of motion. This expression means that the position at time t (left-hand side of the 'maps to' arrow) is $\mathbf{r}_0(t)$ (right-hand side of the arrow). That seems redundant (and is), but it will be helpful in a moment. I have put a subscript '0' on the $\mathbf{r}$ to emphasize that this is a *particular* solution of the equation of motion.

The desire is to have $\tau \mapsto \lambda\mathbf{r}_0(t)$, also be a solution, where $\tau = \lambda^z t$. Making the substitution gives

$$\tau \mapsto \lambda\mathbf{r}_0(\lambda^{-z}\tau).$$

This could equally be written with t in place of τ since it's a dummy argument. So, said in yet another way,

$$\mathbf{r}_1(t) = \lambda\mathbf{r}_0(\lambda^{-z}t)$$

is also a solution of the equation of motion. Now it is clear what the assertion is. If we videotape the motion $\mathbf{r}_0(t)$ and then watch it magnified by a factor λ but run the tape at a speed λ^{-z} times normal, we have no way of knowing that we aren't watching a legitimate motion of the system in real time and actual size. This means that periods are stretched by λ^{-z} (run the tape slow and things take longer to happen).

(d) Deduce Kepler's third law without the constants.

Having thoroughly analyzed what the result in (a) was all about, this is now easy. The Kepler problem is $\alpha = -1$. In that case, $z = 1 - (-1)/2 = 3/2$. So, an orbit scaled up by a factor λ has a period scaled by $\lambda^{3/2}$. I.e. $\tau \propto a^{3/2}$.

2. $|\mathbf{r}|^2$ potential.

A particle of mass m moves in two dimensions in a potential $U(\mathbf{r}) = k|\mathbf{r}|^2/2$.

The equation of motion may be found using Lagrangian methods, or we can do it the good-old Newtonian way. Let's do the latter for a change. Since $\partial|\mathbf{r}|^2/\partial x = x$ etc, the force on the particle is $-\nabla U = -k\mathbf{r}$, leading to an equation of motion

$$m\ddot{\mathbf{r}} = -k\mathbf{r}.$$

Breaking this into x and y components yields a pair of ordinary simple harmonic oscillator equations

$$\begin{aligned} \ddot{x} + \frac{k}{m}x &= 0, \\ \ddot{y} + \frac{k}{m}y &= 0. \end{aligned}$$

We can solve these in our sleep.

$$\begin{aligned} x(t) &= A\cos\omega t + B\sin\omega t, \\ y(t) &= C\cos\omega t + D\sin\omega t, \end{aligned}$$

using $\omega = \sqrt{k/m}$. Solving for the initial position and velocity,

$$\begin{aligned} A &= x(0) \qquad & \omega B &= \dot{x}(0) \\ C &= y(0) \qquad & \omega D &= \dot{y}(0). \end{aligned}$$

Putting it all together,

$$\begin{aligned} x(t) &= x(0)\cos\omega t + \frac{\dot{x}(0)}{\omega}\sin\omega t, \\ y(t) &= y(0)\cos\omega t + \frac{\dot{y}(0)}{\omega}\sin\omega t. \end{aligned}$$

These could be recombined into a single vector equation if we like:

$$\mathbf{r}(t) = \mathbf{r}(0)\cos\omega t + \frac{\dot{\mathbf{r}}(0)}{\omega}\sin\omega t.$$

If all the initial data, $\mathbf{r}(0)$ and $\dot{\mathbf{r}}(0)$ are scaled up by some factor, the orbit still repeats itself after a time $\tau = 2\pi/\omega$, so the period is independent of the amplitude, in complete agreement with problem 1.

The particle cannot travel outward from (or inward toward) the origin forever. There must be a moment when $\dot{r} = 0$, so that $\dot{\mathbf{r}} \cdot \mathbf{r} = 0$. There is nothing to prevent us from choosing that moment as our origin of time, so do it, and then orient the axes so that $\mathbf{r}(0)$ is along x. This reduces the equation for the orbit to

$$\mathbf{r}(t) = \hat{\mathbf{e}}_x\, x(0) \cos \omega t + \hat{\mathbf{e}}_y\, \frac{\dot{y}(0)}{\omega} \sin \omega t.$$

Extracting the x and y components of this,

$$\left(\frac{x(t)}{x(0)}\right)^2 + \left(\frac{y(t)}{\dot{y}(0)/\omega}\right)^2 = 1.$$

This is the equation for an ellipse with semiaxes $x(0)$ and $\dot{y}(0)/\omega$.

3. weighing the sun and the earth.

Use Kepler's third law to find the masses of the sun and the earth. Since $k = GMm$ for these problems and in either case one body is much more massive than the other so that $\mu = Mm/(M + m) = m$ to a good approximation, $k/\mu = GM$. Then, rearranging Kepler's third law a bit,

$$M = \frac{4\pi^2}{G} \frac{a^3}{\tau^2}.$$

Notice that for circular orbits ($a = r$) this follows very easily from $GMm/r^2 = mv^2/r$.

(a) Find the mass of the sun. $G = 6.67 \times 10^{-11}$ N m^2/kg^2, $a = 1.49 \times 10^8$ km. $\tau = 1$ yr $= 3.1557 \times 10^7$ s.

$$M_{\text{sun}} = 5.919 \times 10^{11} \frac{(1.49 \times 10^{11})^3}{(3.1557 \times 10^7)^2}\,\text{kg} = 1.97 \times 10^{30}\,\text{kg}$$

(b) Find the mass of the moon. $\tau = 27.3$ days $= 2.359 \times 10^6$ s. $a = 3.8 \times 10^5$ km.

$$M_{\text{earth}} = 5.919 \times 10^{11} \frac{(3.8 \times 10^8)^3}{(2.359 \times 10^6)^2}\,\text{kg} = 5.84 \times 10^{24}\,\text{kg}$$

Given that the radius of the earth is 6380 km, find the mean density of the earth.

$$\rho = \frac{M_{\text{earth}}}{4\pi R^3/3} = 5.37 \times 10^3\,\text{kg/m}^3 = 5.37\,\text{g/cm}^3$$