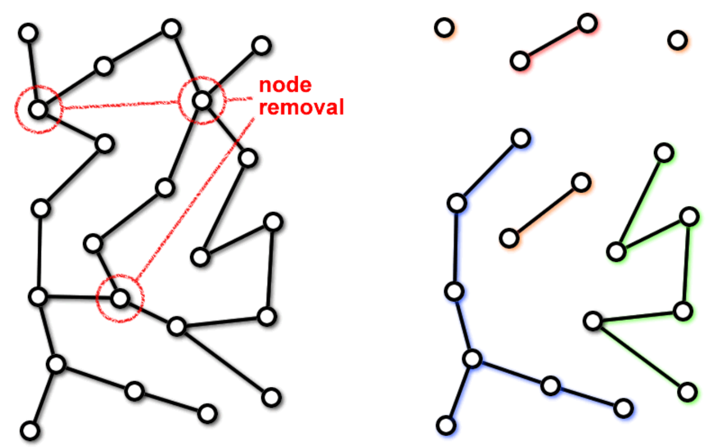
Topological perturbation of complex networks

Perturbations in complex systems can deactivate some of the edges or nodes.
Edge loss: the edge is deleted
Node loss: the node and all its edges are deleted

Effects on the global topology:

- increase of path lengths,
- separation into isolated clusters.



More connected network - less effect of an edge removal
But bridges are definite points of vulnerability!
The effect of a node removal depends on the number and characteristics of its edges.

Resilience to perturbations

Topological resilience: the remaining nodes are still connected.
the average distance does not increase.

We will remove edges/nodes one by one, and look at

- the size of the giant connected component
- the average distance between nodes in the giant connected component

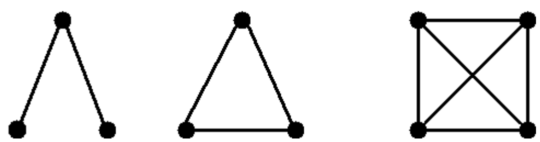
Influencing factors: the type of removal
the original topology

Evolution of a random graph

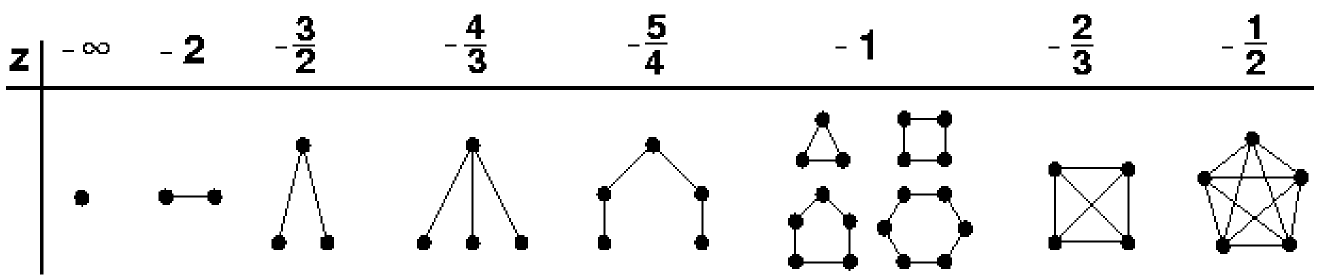
Assume that the connection probability is a power-law of N, $p = cN^z$

Assume that z increases from $-\infty$ to 0

Look for trees, cycles (circuits) and cliques in the graph.



Appearance thresholds: $p \sim N^z$

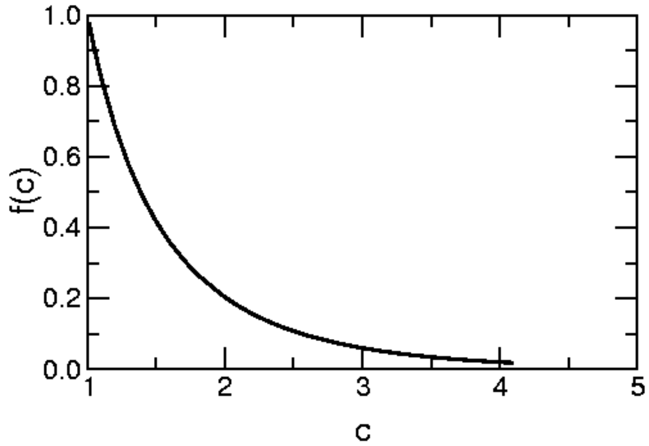


The graph contains cycles of any length if $z \geq -1$

Clusters in a random graph

- If $\lim_{N\to\infty} pN = 0$ the graph contains only isolated trees.
- If $p = cN^{-1}$ with $c < 1$ the graph has isolated trees and cycles.
- At $p = cN^{-1}$ with $c = 1$ a giant cluster appears.
- The size of the giant cluster approaches N rapidly as c increases.

$$S = (f(1) - f(c))N$$



- The graph becomes connected if $\lim_{N\to\infty} \frac{p}{\ln N / N} = \infty$

Existence of a giant connected cluster in a general random graph

average size of clusters $\langle s \rangle = 1 + \frac{G_0'(1)}{1 - G_1'(1)}$

$G_0(x)$ – node degree generating function

$G_1(x)$ – generating function for the degree of an edge endpoint

$$G_0(x) = \sum_{k=0}^{\infty} P(k) x^k$$

$$G_1(x) = \frac{\sum_k k P(k) x^{k-1}}{\sum_k k P(k)}$$

$\langle s \rangle \to \infty$ when

$$G_1'(1) \equiv \frac{\sum_k k P(k)}{\sum_k k(k-1) P(k)} = 1$$

equivalent to

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = 2$$

A giant connected component exists if the graph is sufficiently heterogeneous.

Edge removal in random graphs

Start with a connected ER random graph with conn. prob. *p.*

$$\lim_{N\to\infty} \frac{p}{\ln N / N} = \infty$$

Remove a random fraction *f* of the edges.

Expected result: an ER graph with conn. prob. *p(1-f)*

Connected if $$\lim_{N\to\infty} \frac{p(1-f)}{\ln N / N} = \infty$$

For a broad class of starting graphs, there exists a threshold edge removal probability such that if a smaller fraction of edges is removed the graph is still connected.

B. Bollobas, Random Graphs, 1985

Node removal

Removing a node deactivates all its edges.

We can expect that the effect of the node removal will depend on the number of edges it had.

The size of the connected component will decrease at least by one.

Assume we have two networks with the same number of nodes and edges, and remove a fraction *f* of the nodes.

Can the two networks' resilience be different?

Numerical simulations of network resilience

Two networks with equal number of nodes and edges

- ER random graph
- scale-free network (BA model)

Study the properties of the network as an increasing fraction *f* of the nodes are removed.

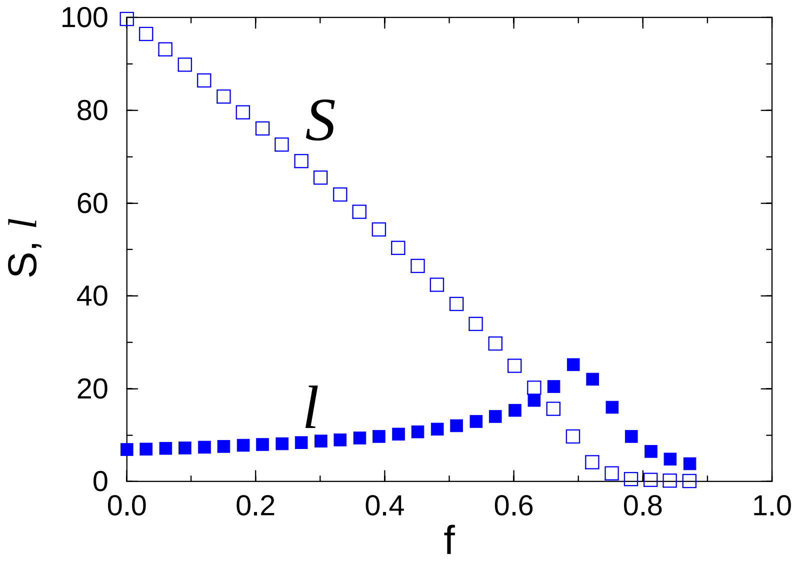
Node selection: random (errors)
the node with the largest number of edges (attack)

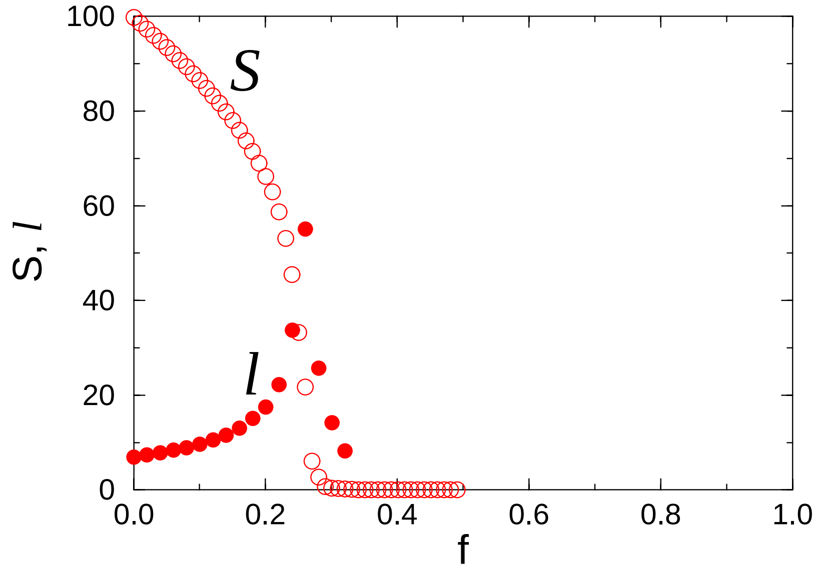
Measures: the fraction of nodes in the largest connected cluster, S
the average distance between nodes in the largest cluster, *l*

R. Albert, H. Jeong, A.-L. Barabási, Nature 406, 378 (2000)

Random networks respond similarly to errors and attacks

The size *S* and average path length *l* of the largest cluster





Random failure

Targeted attack

Similar to an inverse graph evolution.

Breakdown transition in general random graphs

Consider a random graph with arbitrary $P(k_0)$

A giant cluster exists if each node is connected to at least two other nodes.

$$\frac{\langle k^2 \rangle}{\langle k \rangle} = 2$$

Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).

After the random removal of a fraction *f* of the nodes,

$$\langle k \rangle = \langle k_0 \rangle (1-f), \quad \langle k^2 \rangle = \langle k_0^2 \rangle (1-f)^2 + \langle k_0 \rangle f(1-f)$$

Breakdown threshold:

$$f_c = 1 - \frac{1}{\frac{\langle k_0^2 \rangle}{\langle k_0 \rangle} - 1}$$

Application: random graphs

Consider a random graph with connection probability *p* such that at least a giant cluster is present in the graph.

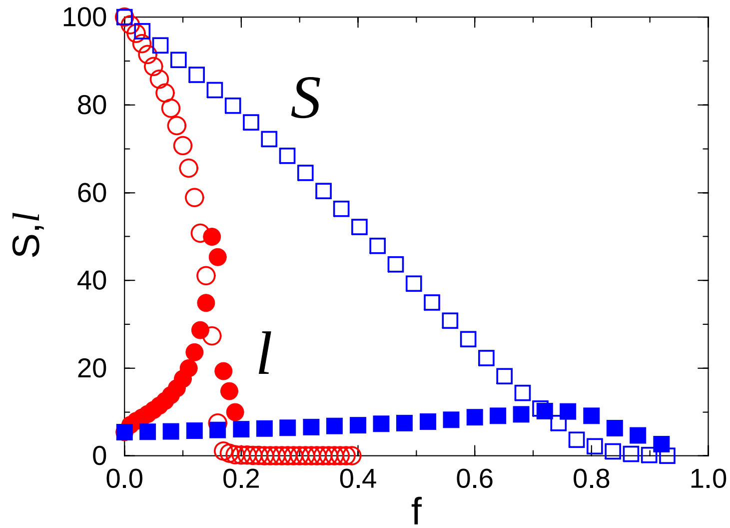
$$\langle k_0 \rangle = pN, \quad \langle k_0^2 \rangle = (pN)^2 + pN$$

Find the critical fraction of removed nodes such that the giant cluster is destroyed.

$$f_c = 1 - \frac{1}{\frac{\langle k_0^2 \rangle}{\langle k_0 \rangle} - 1} = 1 - \frac{1}{pN}$$

The higher the original average degree, the larger damage the network can survive.

Scale-free networks are more error tolerant, but also more vulnerable to attacks



- blue symbols: random failure
- red symbols: targeted attack

Attacks: same breakdown scenario as for random graphs.
Failures: little effect on the integrity of the network.

Is the low peak in average distance a finite size effect?

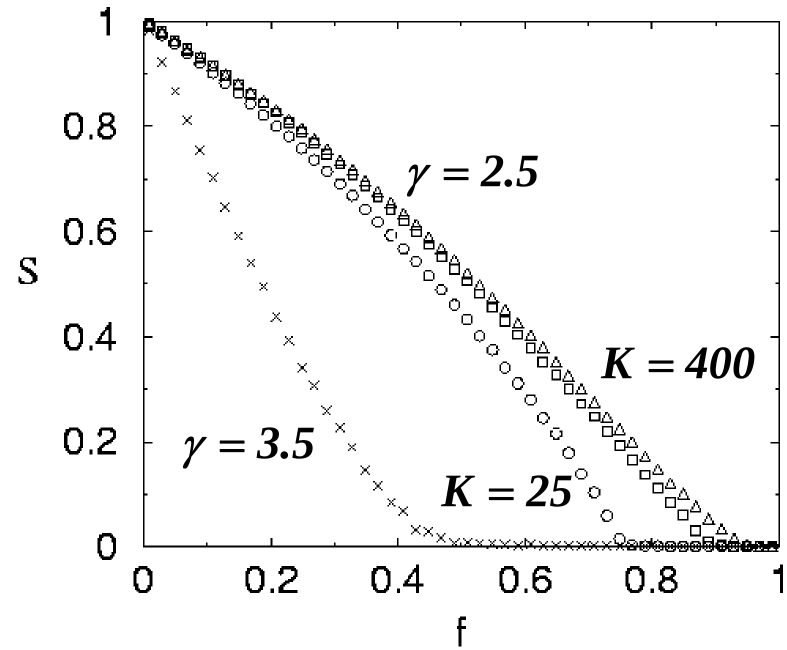
How does the breakdown threshold depend on the size of the network and the degree exponent?

Breakdown threshold of scale-free random graphs

Scale-free random graph with

$$P(k) = Ak^{-\gamma}, \text{ with } k = m, \ldots K$$

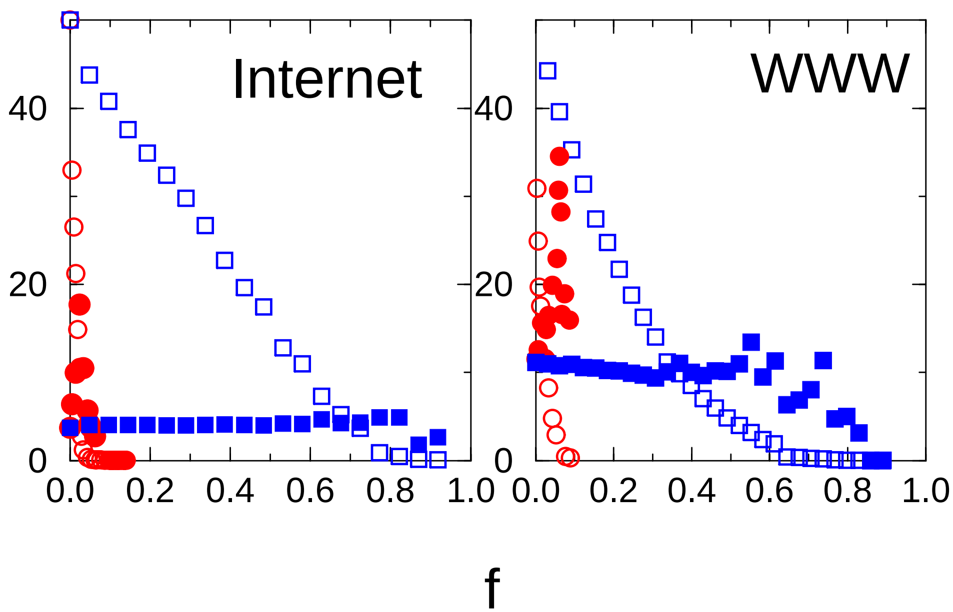
$$f_c = 1 - \frac{1}{\frac{\gamma - 2}{\gamma - 3} m - 1}$$



Infinite systems with $\gamma < 3$ do not break down under random failure.

Cohen et al., Phys. Rev. Lett. 85, 4626 (2000).

Real scale-free networks show the same dual behavior

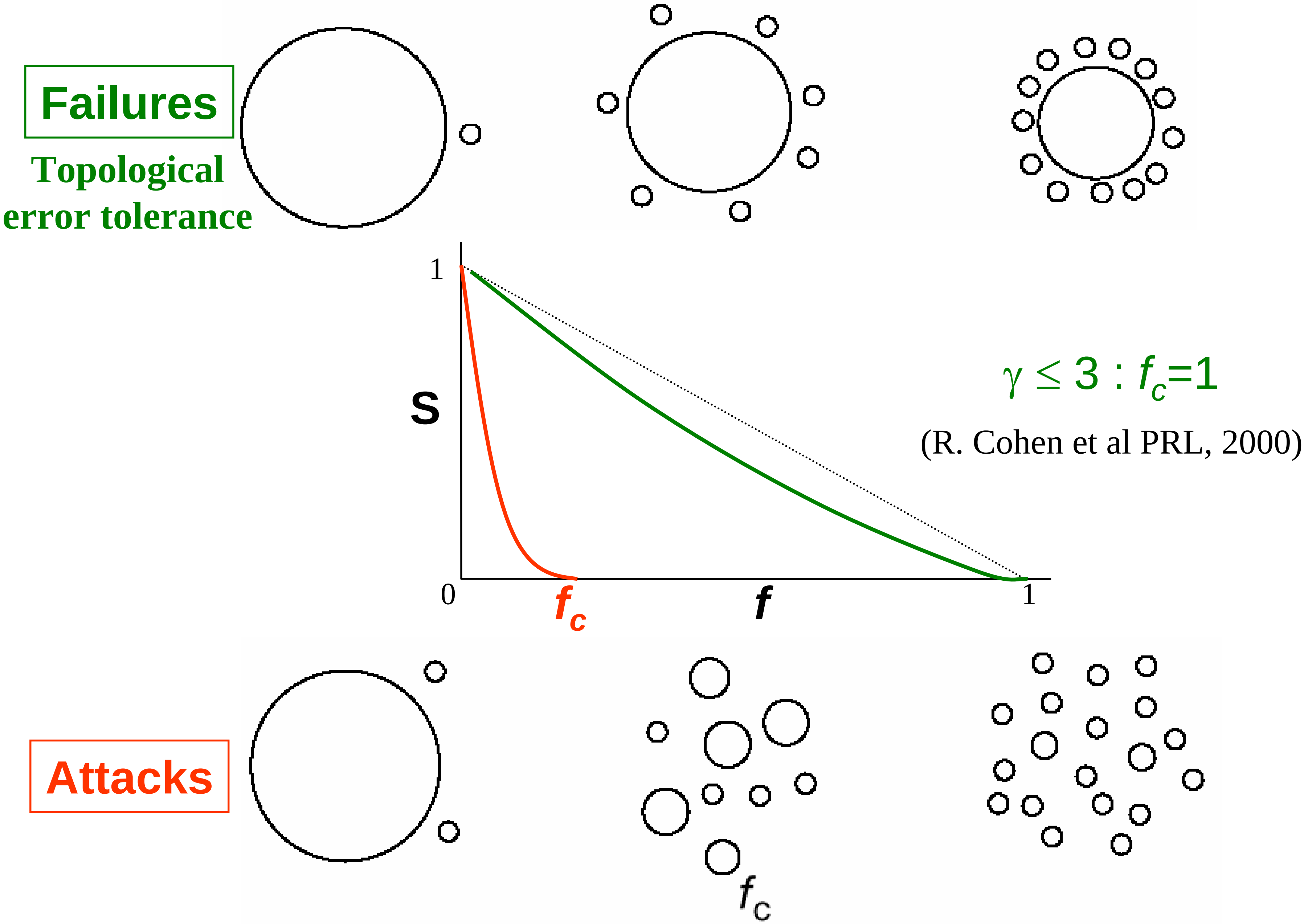


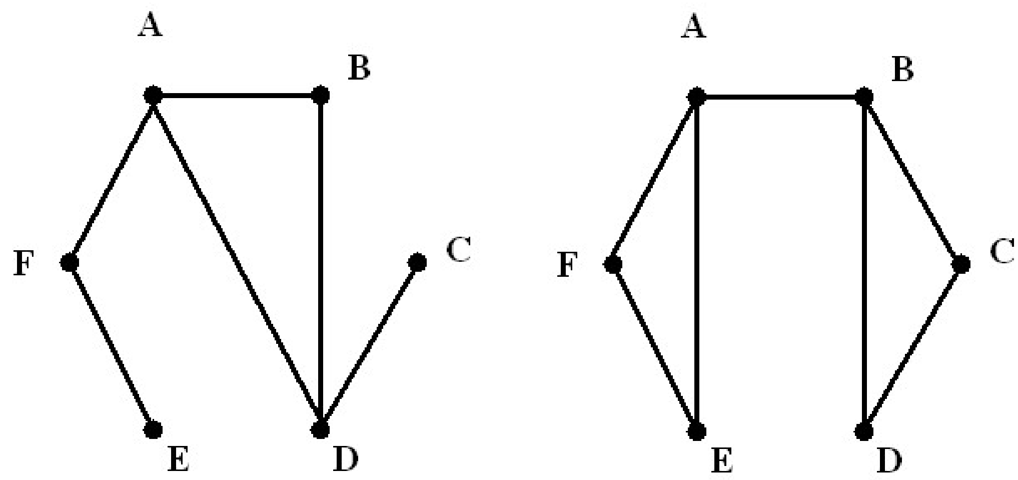
- break down if 5% of the nodes are eliminated selectively
- resilient to the random failure of 50% of the nodes.

- blue symbols: random failure
- red symbols: targeted attack

Similar results have been obtained for metabolic networks and food webs.

Robustness of scale-free networks





1. Rank order the nodes by the effect of their removal. What were your criteria in doing so?

2. For each node, determine what is the effect of its removal on the size of the connected component, and the distances on the connected component.

3. Do the results match your expectations?

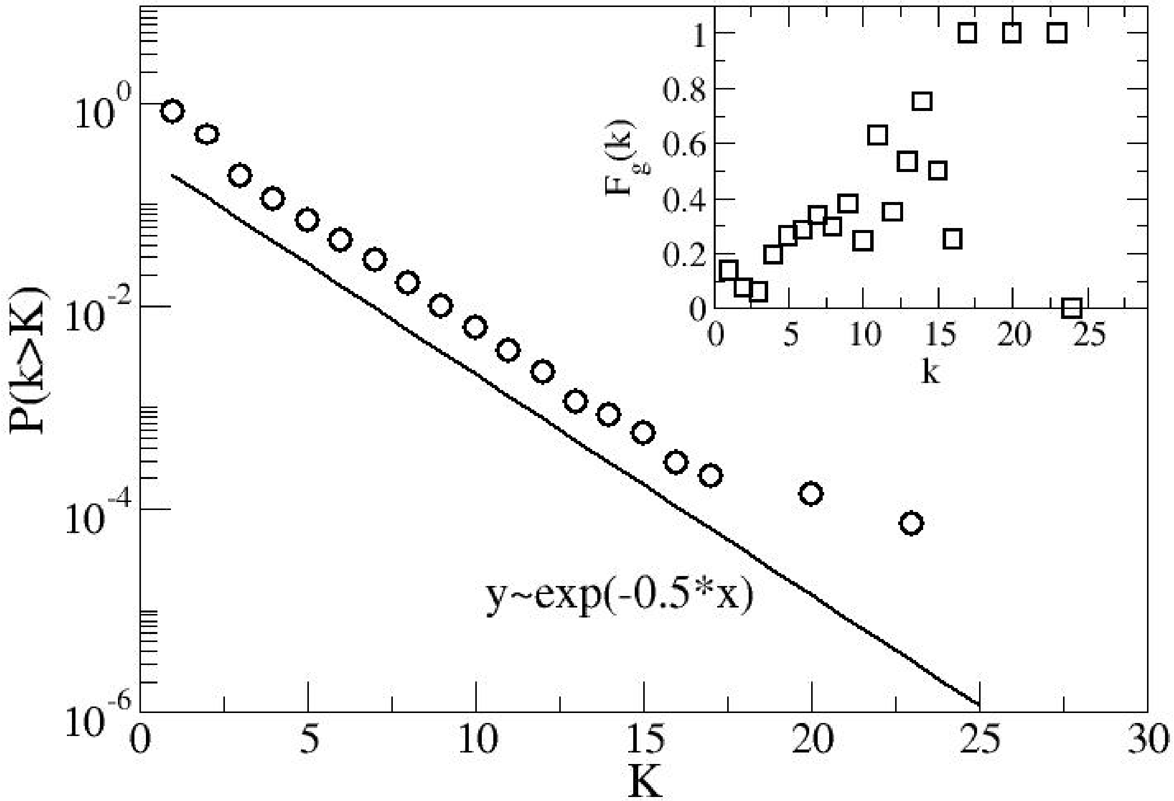
Case study: NA powergrid

- Nodes: generators, transmission substations, distribution substations
- Edges: high-voltage transmission lines

- 14099 nodes: 1633 generators, 2179 distribution substations, the rest transmission substations
- 19,657 edges

- The role of the power grid is to route power from generators to distribution substations (and then to customers)
- Connected network: power from any generator is in principle accessible to any substation

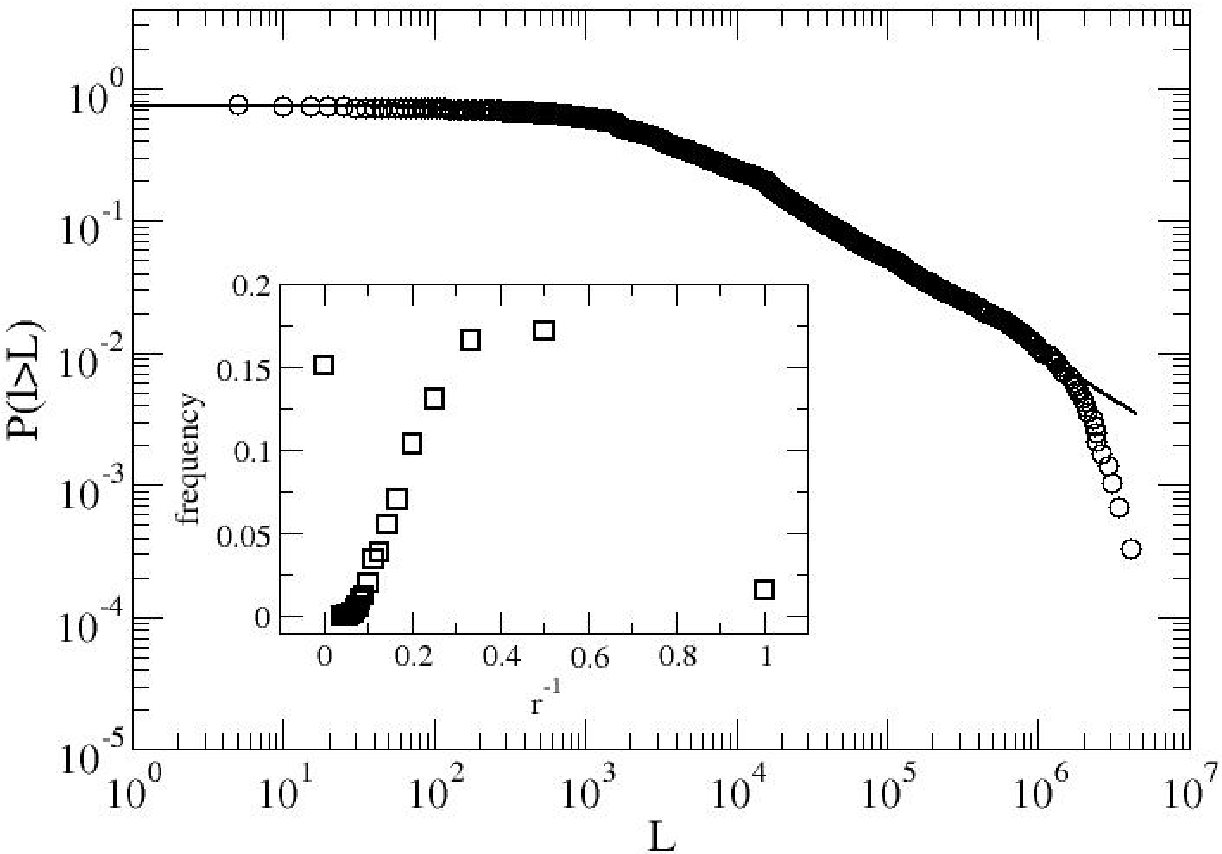
Degree distribution



$$P(k > K) \approx \exp(-0.5K)$$

Betweenness distribution and edge range

Edge range: what would the distance between the endpoints of an edge be if the edge is removed.



$$P(l > L) \approx (2500 + L)^{-0.7}$$

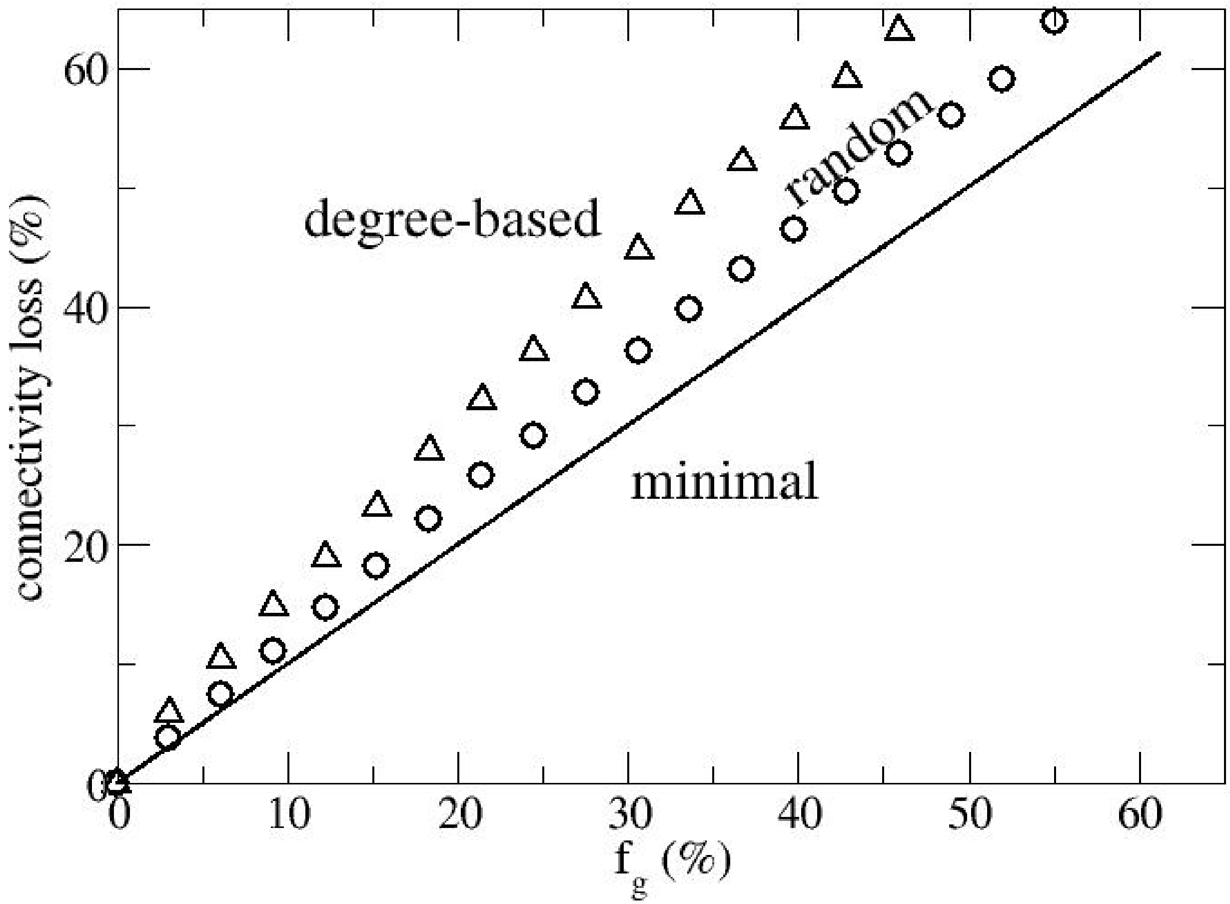
Resilience of the NA powergrid

- The relevant question is whether distribution substations receive enough power
- Studied measure: how many generators can feed a given distribution substation
- Average connectivity – the fraction of generators able to feed a given substation, averaged over substations

$$Co = \left\langle \frac{N_g^i}{N_g} \right\rangle_i$$

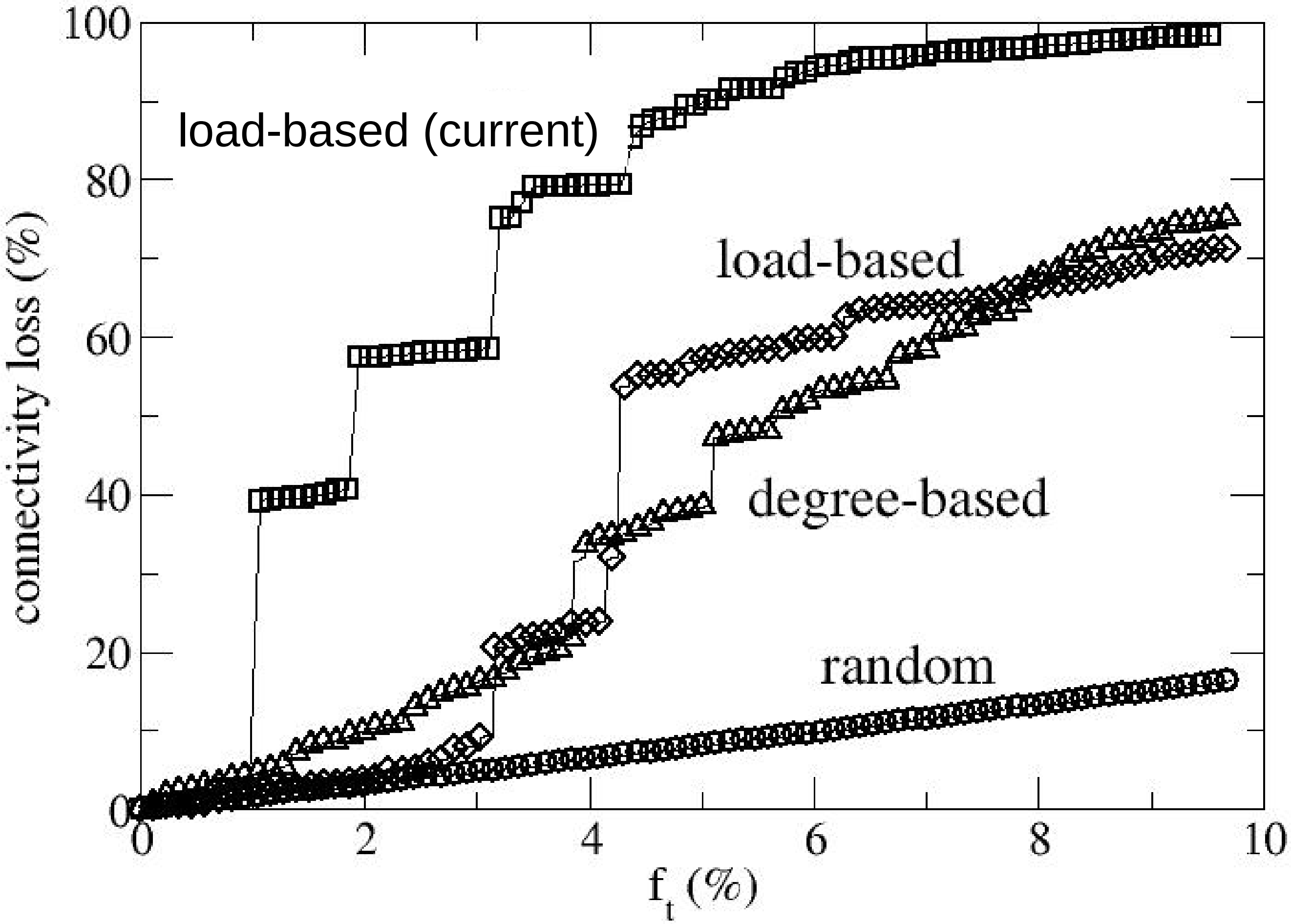
- Connectivity loss $CL = 1 - \left\langle \frac{N_g^i}{N_g} \right\rangle_i$ expressed as a percentage
- Generator removal will definitely lead to connectivity loss, transmission substation removal not necessarily.

Connectivity loss for generator removal



R. Albert, I. Albert, G.N. Nakarado, Phys. Rev. E (2004)

Connectivity loss for transmission substation removal



Highest damage if the next substation to be removed is the current highest-load substation

Limitations of topological resilience

- The most relevant measure of connectivity may not be the size of the giant connected cluster
- The effects of removing a node or edge propagate through the network
  - E.g. cascading failure on the power grid, gene mutation
  - Depends on the dynamical properties of the network
- The network topology still determines the boundaries of propagating failure