

Network models

Properties common to many large-scale networks, independently of their origin and function:

1. The degree and betweenness distribution are decreasing functions, usually power-laws. **scale - free**
2. The distances scale logarithmically with the network size

$$l \approx \frac{\log N}{\log \langle k \rangle} \quad \text{small world}$$

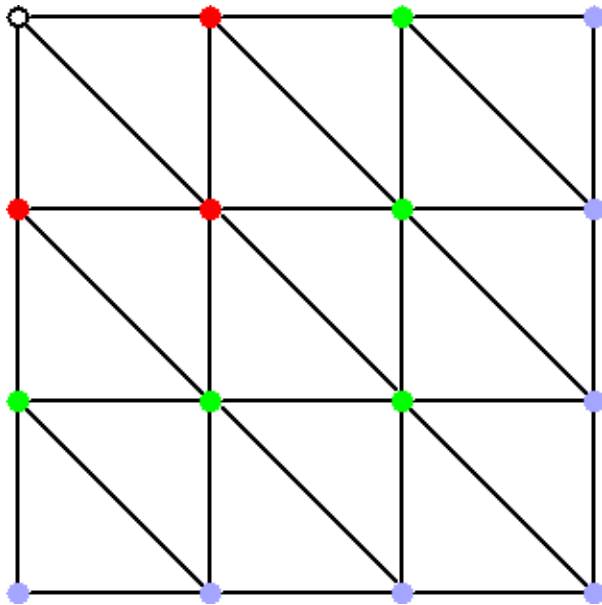
3. The clustering coefficient does not seem to depend on the network size, and is larger than the clustering coefficient of comparable random graphs

There are two model families proposed to explain these properties:

Small world network models and **scale-free network models**.

Benchmark 1: regular lattices

One-dimensional lattice: $l \approx N, k = \text{const}, C = \text{const}$



$$l \approx L \approx N^{1/2}$$

$k = 6$ for inside nodes

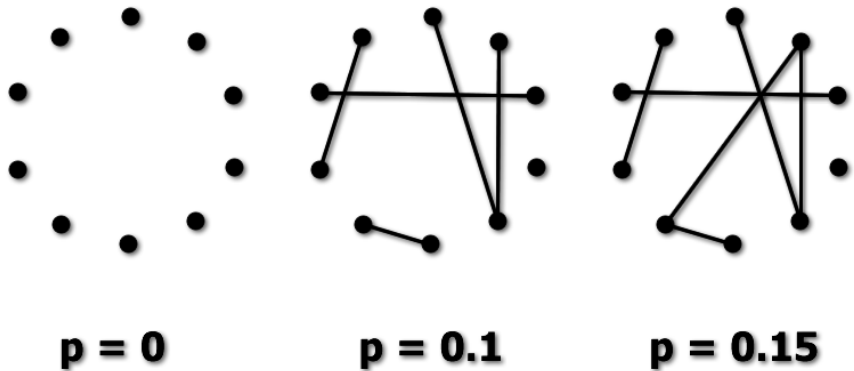
$$C = \frac{6}{15} \text{ for inside nodes}$$

The average path-length varies as $l \approx N^{1/D}$

Constant degree (coordination number), constant clustering coefficient.

Benchmark 2: random graph theory

Erdős-Rényi algorithm - [Publ. Math. Debrecen 6, 290 \(1959\)](#)



- fixed node number N
- connecting pairs of nodes with probability p

Expected degree distribution: $P_{rand}(k) \cong C_{N-1}^k p^k (1-p)^{N-1-k}$

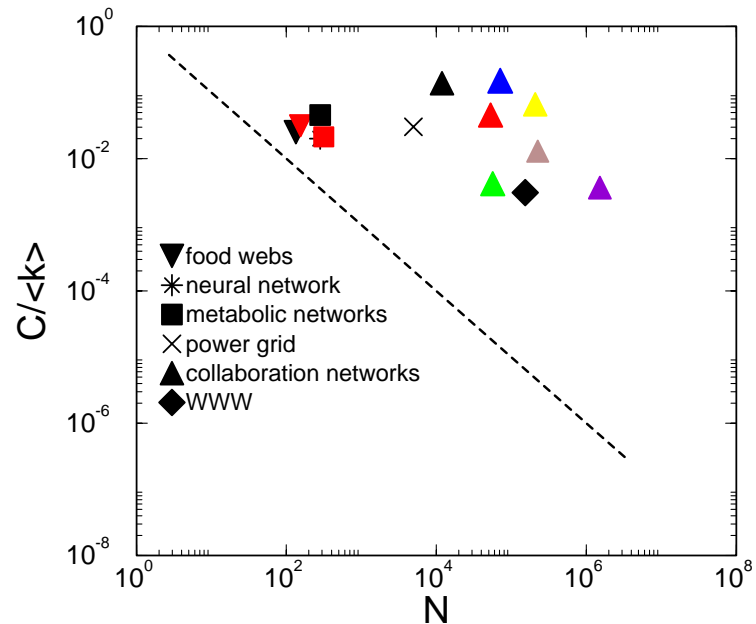
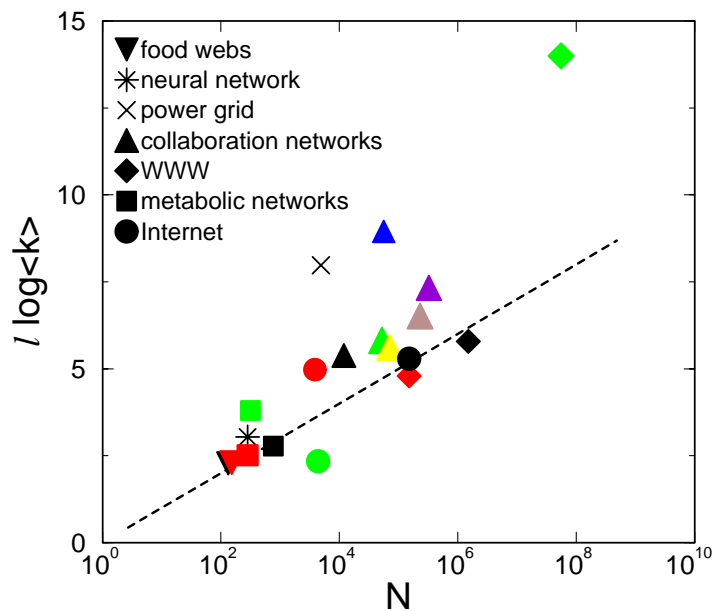
Expected path length: $l_{rand} \approx \frac{\log N}{\log \langle k \rangle}$

Expected clustering coefficient: $C_{rand} = p = \frac{\langle k \rangle}{N}$

Path length and order in real networks

$$l_{rand} = \frac{\log N}{\log \langle k \rangle}$$

$$C_{rand} = \frac{\langle k \rangle}{N}$$

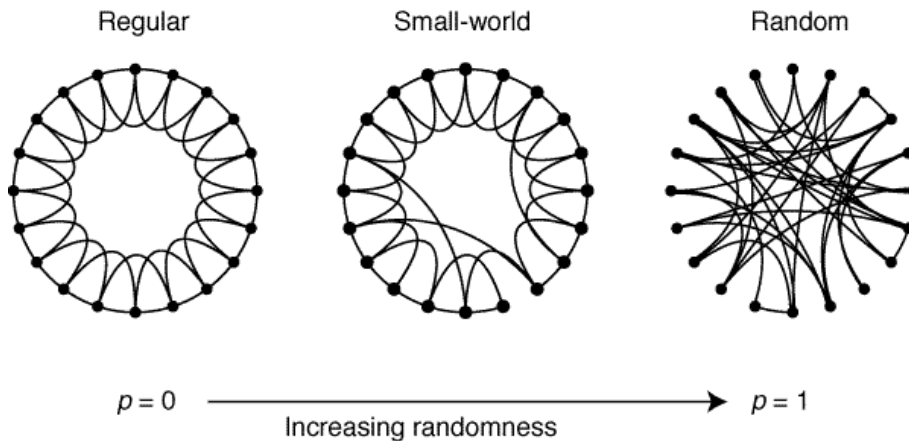


Real networks have short distances like random graphs yet show signs of local order.

Small-world networks

Real networks resemble both regular lattices and random graphs – perhaps they are in between.

Watts-Strogatz model - [D. Watts, S. Strogatz, Nature 393, 440 \(1998\)](#)

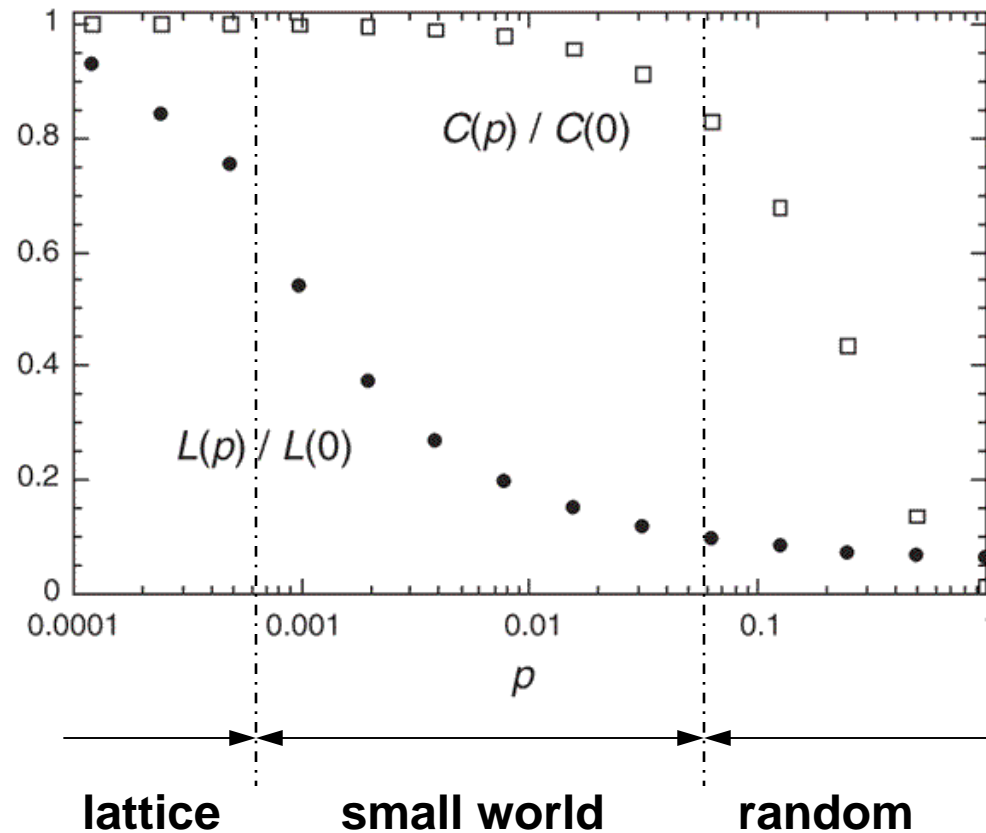


- lattice with K neighbors
- rewire edges with probability p

$$l = \frac{N}{2K}, C = \frac{3(K-2)}{4(K-1)} \quad \longrightarrow \quad l \approx \frac{\log N}{\log K}, C \approx \frac{K}{N}$$

Is there a regime with small l and large C ?

Transition from a lattice to a small world



There is a broad interval of p over which $C(p) \cong C(0)$ but $l(p) \cong l(1)$

The onset of the small-world behavior depends on the system size

$$l(N, p) \approx \frac{N^{1/d}}{K} f(pKN)$$

d is the dimension of the lattice

$$f(u) = \begin{cases} \text{const} & \text{if } u \ll 1 \\ \ln u / u & \text{if } u \gg 1 \end{cases}$$

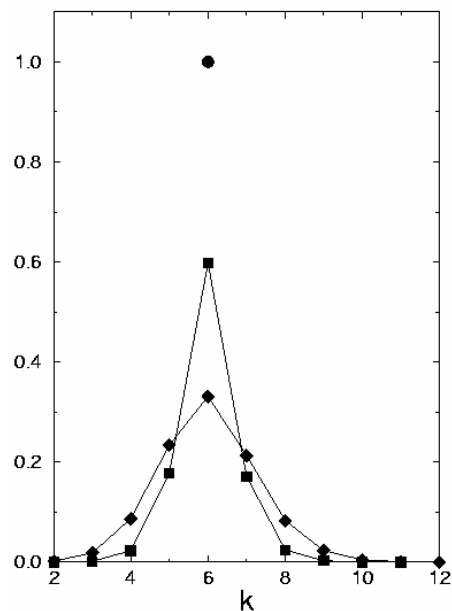
lattice - like
random graph - like

The transition point depends on the rewiring probability, the size of the network and the average degree.

$$C(p) = C(0)(1-p)^3$$

These results cannot be directly compared to most real networks because the rewiring probability p is not known.

Degree distribution of a small-world network



Rewiring does not change the average degree, but modifies the degree distribution.

$$\langle k \rangle = K$$

$P(k)$ depends on the rewiring parameter p , but is always centered around $\langle k \rangle$.


Degree distribution similar to that of a random graph, with exponentially small probability for very highly connected nodes.

Ex. 1

A variant of the Watts-Strogatz model adds random edges to a regular lattice. Start with a 1D lattice where every node has degree K . For each existing edge of a node, add an edge with a probability p . The endpoint of the edge is selected randomly from all other nodes. How many edges do you expect the graph will have after edge addition?

Ex. 2

How do you expect the degree distribution will look like after edge addition? Will it be symmetrical or not?

$$k_i = K + m_i + n_i$$


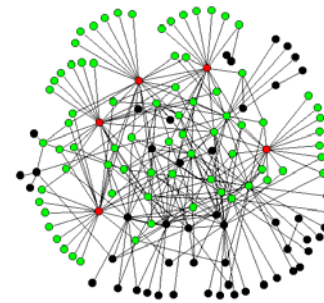
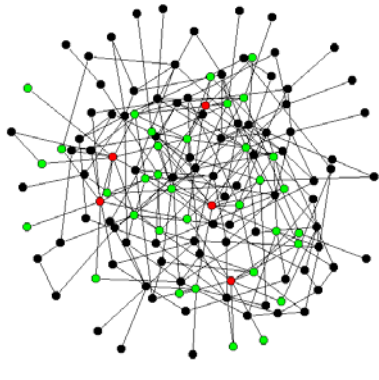
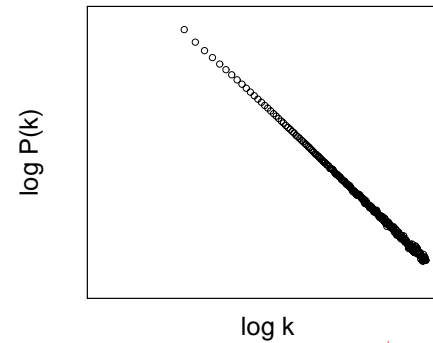
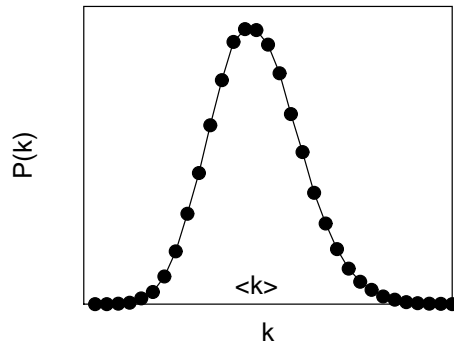
starting point endpoint

$$P(m) \cong C_{K/2}^m p^m (1-p)^{K/2-m}$$

$$P(n) \cong C_{pNK/2}^n \left(\frac{1}{N}\right)^n \left(1 - \frac{1}{N}\right)^{pNK/2-n}$$

Minimum: K , peak: $K+pK$

The scale-free degree distribution indicates a heterogeneous topology



New models are needed to reproduce this feature.

We need to uncover the mechanisms responsible for the scale-free $P(k)$

- random graphs
- small-world networks
- scale-free random graphs

Static (#of nodes fixed)

Real networks continuously change

- random graphs
- small-world networks

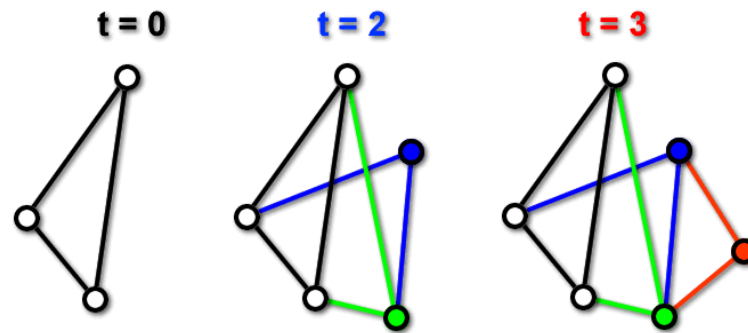
Homogeneous

Scale-free degree distribution - the nodes are not equivalent

We need to model the evolution of networks, not just their topology.

A simple model of network assembly and evolution (BA model)

Start with a small seed of m_0 nodes and $m_0(m_0-1)/2$ edges.



- **growth:** a node and m edges added at every step
- **preferential attachment:** $\Pi(k_i) = \frac{k_i}{\sum_j k_j}$

Price, J. Amer. Soc. Inform. Sci. 27, 292 (1976)

Barabási and Albert, Science 286, 509 (2000)

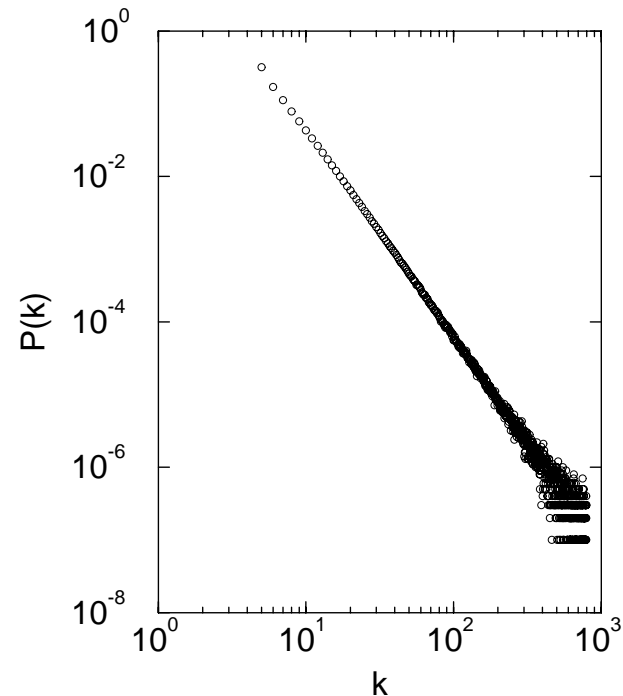
General properties of the network

- nr. of nodes: $N = m_0 + t$
- nr. of edges: $E = \frac{m_0(m_0 - 1)}{2} + m t$

- average degree: $\langle k \rangle = \frac{2E}{N} \rightarrow 2m$

- degree distribution:

$$P(k) \xrightarrow{t \rightarrow \infty} A k^{-3}$$



Although the network grows, the degree distribution becomes stationary.

Analytical determination of P(k)

The degree of “old” nodes increases by acquiring new edges. The probability of an old node with degree k_i receiving a new edge is

$$m\Pi(k_i) = m \frac{k_i}{\sum_j k_j} \approx \frac{k_i}{2t}$$

Degree increase: $\Delta k_i = \begin{cases} 1 & \text{with prob. } \frac{k_i}{2t} \\ 0 & \text{otherwise} \end{cases}$

Choices:

follow the increase in the number of nodes with degree k_i (rate equation approach)

follow the increase in time in k_i (continuum theory)

Rate equation approach

Change in average number of nodes with degree k

$$\frac{dN_k}{dt} = m \frac{(k-1)N_{k-1}(t) - kN_k(t)}{\sum_k kN_k(t)} + \delta_{k,m}$$

$k-1 \rightarrow k$ $k \rightarrow k+1$
 (Arrows point from these labels to the terms $(k-1)N_{k-1}(t)$ and $-kN_k(t)$ in the numerator.)
 ← first node (points to $\delta_{k,m}$)
 number of edges of new node (points to m)
 normalization (points to $\sum_k kN_k(t)$)

$$P(k) = N_k(t)/N = \lim_{t \rightarrow \infty} N_k(t)/t$$

Plug in:

$$P(k) = m \frac{(k-1)P(k-1) - kP(k)}{\sum_k kP(k)} + \delta_{k,m}$$

$\langle k \rangle$ (points to $\sum_k kP(k)$)

Degree distribution

The rate equation leads to a recursive relationship between $P(k)$ and $P(k-1)$

$$P(k) = \frac{(k-1)P(k-1) - kP(k)}{2} + \delta_{k,m}$$

$$P(k) = \begin{cases} \frac{k-1}{k+2} P(k-1) & \text{for } k > m \\ \frac{2}{m+2} & \text{for } k = m \end{cases}$$

Leads to
$$P(k) = \frac{2m(m+1)}{k(k+1)(k+2)} \approx k^{-3}$$

Stationary power law with an exponent $\gamma = 3$

P. Krapivsky, S. Redner, F. Leyvraz, Phys. Rev. Lett. 85, 4629 (2000)

Continuum theory

Degree increase:

$$\Delta k_i = \begin{cases} 1 & \text{with prob. } \frac{k_i}{2t} \\ 0 & \text{otherwise} \end{cases}$$

Assume that there is a continuous rate of change of k_i

$$\frac{dk_i}{dt} \approx \frac{k_i}{2t}$$

Initial condition: $k_i(t_i) = m$ t_i - time when node i was added to the network

Solve for k_i

$$k_i(t, t_i) = m \sqrt{\frac{t}{t_i}}$$

Continuum theory

$$k_i(t, t_i) = m \sqrt{\frac{t}{t_i}}$$

Calculate the cumulative probability that $k_i(t, t_i) < k$

$$P(k_i(t) < k) = P_t(t_i > \frac{m^2 t}{k^2}) = 1 - P_t(t_i < \frac{m^2 t}{k^2})$$

(Almost) all t_i values are equiprobable $p(t_i) = \frac{1}{m_0 + t}$

We need to find the probability of t_i being smaller than a given value $T = m^2 t / k^2$ given the distribution $p(t_i)$.

$$P(x < X) = \sum_{x: x < X} p(x)$$

$$P(t_i < T) = \frac{m_0}{m_0 + t} + \sum_{t_i: 0 < t_i < T} \frac{1}{m_0 + t} = \frac{m_0 + T}{m_0 + t}$$

$$P(t_i < \frac{m^2 t}{k^2}) = \frac{m_0 + \frac{m^2 t}{k^2}}{m_0 + t}$$

Continuum theory

$$P(k_i(t) < k) = 1 - P_t(t_i < \frac{m^2 t}{k^2}) = 1 - \frac{m_0 + \frac{m^2 t}{k^2}}{m_0 + t}$$

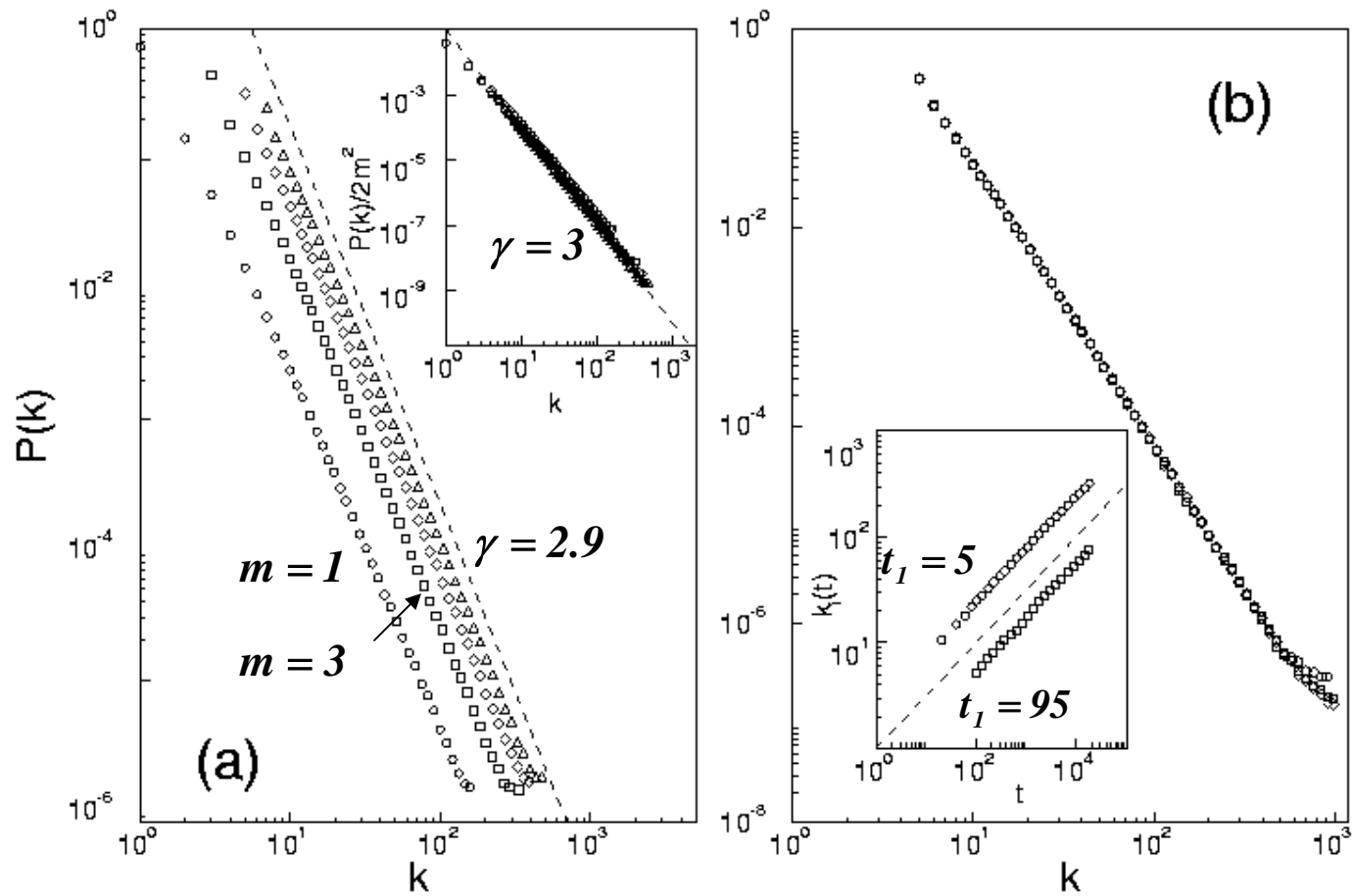
The (non-cumulative) degree distribution is the derivative of the cumulative degree distribution with respect to k

$$P(k) = \frac{dP(k_i(t) < k)}{dk} = \frac{2m^2 t}{m_0 + t} \frac{1}{k^3} \sim k^{-3}$$

Stationary power law with an exponent $\gamma = 3$

The functional form of the degree distribution does not depend on m and m_0 in the long-time limit.

Simulations and theory agree



Ex. 1

Start from a seed of two nodes connected by an edge. At each step, add a new node, and connect it by a single edge to a preexisting node.

Let the probability of selection be directly proportional with the degree of the “old” node. (Is there an easy way to do this?)

Continue growing the graph until you reach 15 nodes. Describe the graph (average degree, degree distribution, clustering coefficient, connectivity, maximum distance).

Ex. 2

How will the properties of the graph change if at each step a new node and two new edges are added?

Model A

growth

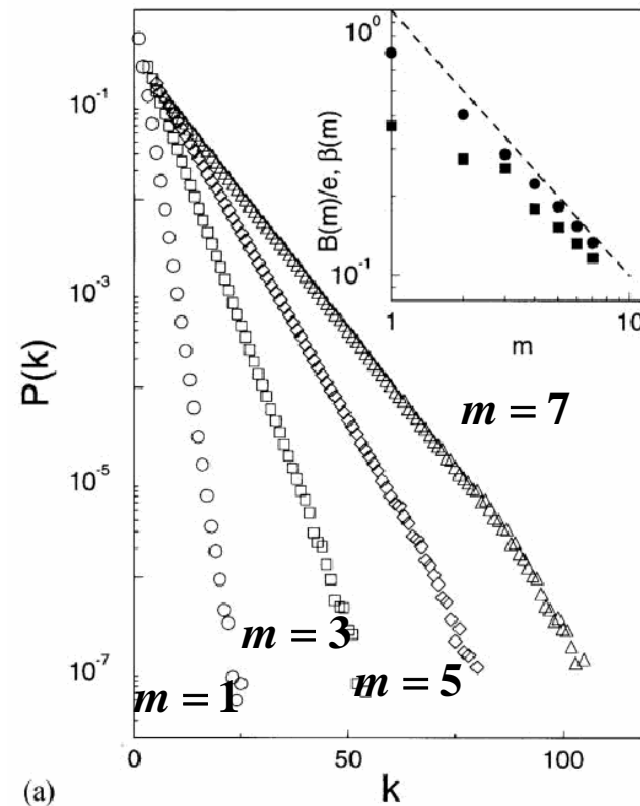
~~preferential attachment~~

$\Pi(k_i)$: uniform

$$\frac{dk_i}{dt} = A\Pi(k_i) = \frac{m}{m_0 + t - 1}$$

$$k_i(t) = m \left(\ln\left(\frac{m_0 + t - 1}{m + t_i - 1}\right) + 1 \right)$$

$$P(k) = \frac{e}{m} \exp\left(-\frac{k}{m}\right)$$



Characteristic degree scale: m

Model B

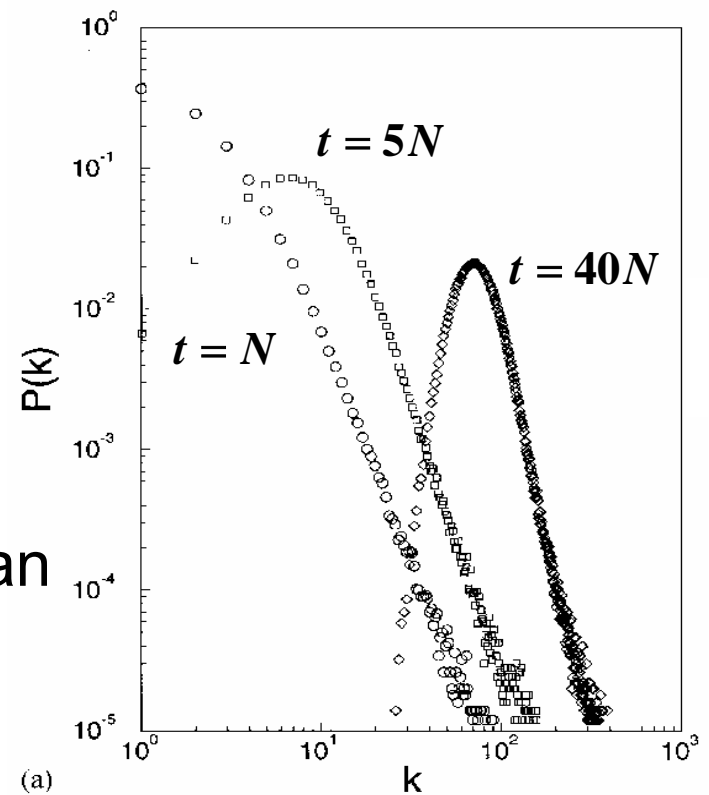
~~growth~~ preferential attachment

$$\frac{dk_i}{dt} = A\Pi(k_i) + \frac{1}{N} = \frac{N}{N-1} \frac{k_i}{2t} + \frac{1}{N}$$

$$k_i(t) = \frac{2(N-1)}{N(N-2)} t + Ct^{\frac{N}{2(N-1)}}$$

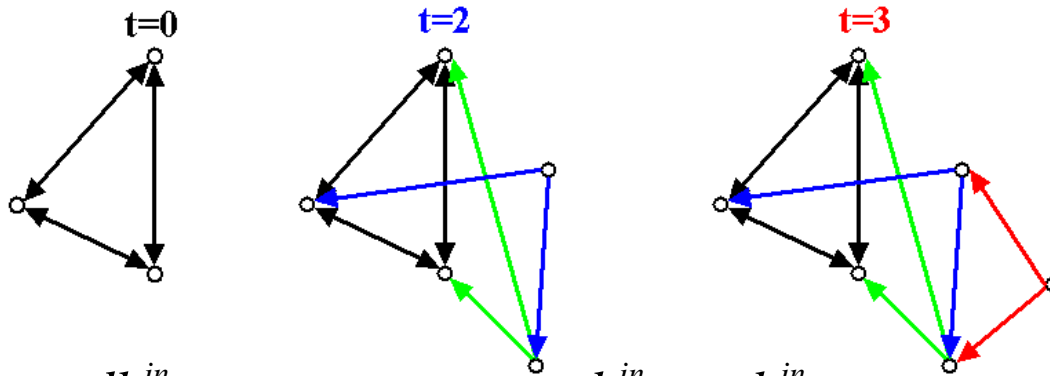
$$\langle k \rangle = \frac{2mt}{N}$$

$P(k)$: power law (initially) \Rightarrow Gaussian



BA algorithm with directed edges

New edges are directed from the new to the old nodes



$$k_i^{out} = m \text{ for } i > m_0$$

k_i^{in} varies

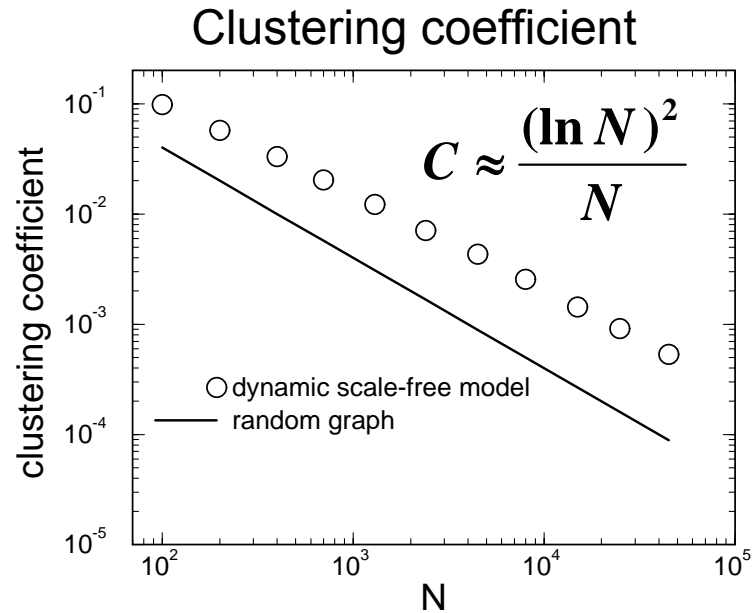
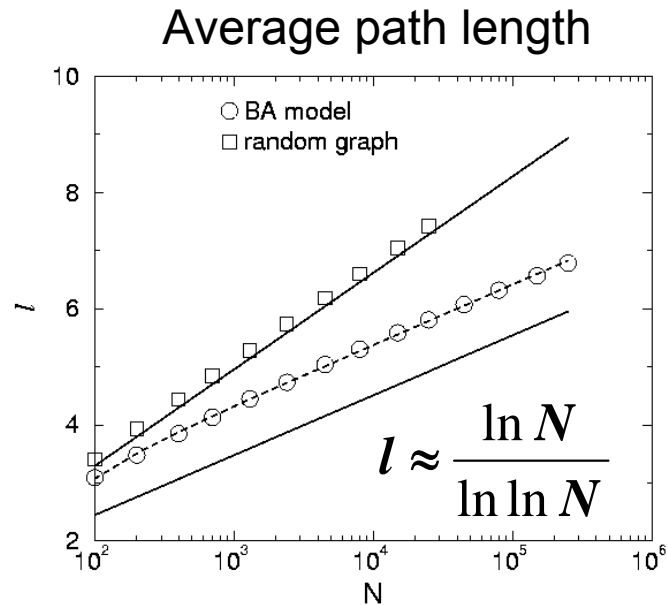
$$\langle k^{in} \rangle = m$$

$$\frac{dk_i^{in}}{dt} = m\Pi(k_i^{in}) = m \frac{k_i^{in}}{\sum_j k_j^{in}} = \frac{k_i^{in}}{t}$$

$$P(k_i^{in}(t) < k) = 1 - \frac{m_0 + \frac{mt}{k}}{m_0 + t} \quad \Rightarrow \quad P_{in}(k) = \frac{mt}{m_0 + t} \frac{1}{k^2} \sim k^{-2}$$

The degree exponent of the directed scale-free network is 2 !

How do the other network measures compare with real networks?



Average distances smaller in the BA model than in equivalent random graphs. but not as small as in scale-free random graphs.

[Cohen et al, in Handbooks of Graphs and Networks \(2003\)](#)

Clustering coefficient decreases with network size.

[B. Bollobás and O. Riordan, in Handbooks of Graphs and Networks \(2003\)](#)

Evolving network models

The scale-free model is only a minimal model.

Makes the simplest assumptions:

- linear growth

$$\langle k \rangle = 2m$$

- proportional preferential attachment

$$\Pi(k_i) \propto k_i$$

Does not capture

- variations in the shape of the degree distribution

- variations in the exponent of the power-law region

- the size-independent clustering coefficient

Hypothesis: the basic mechanisms need to be augmented by the incorporation of

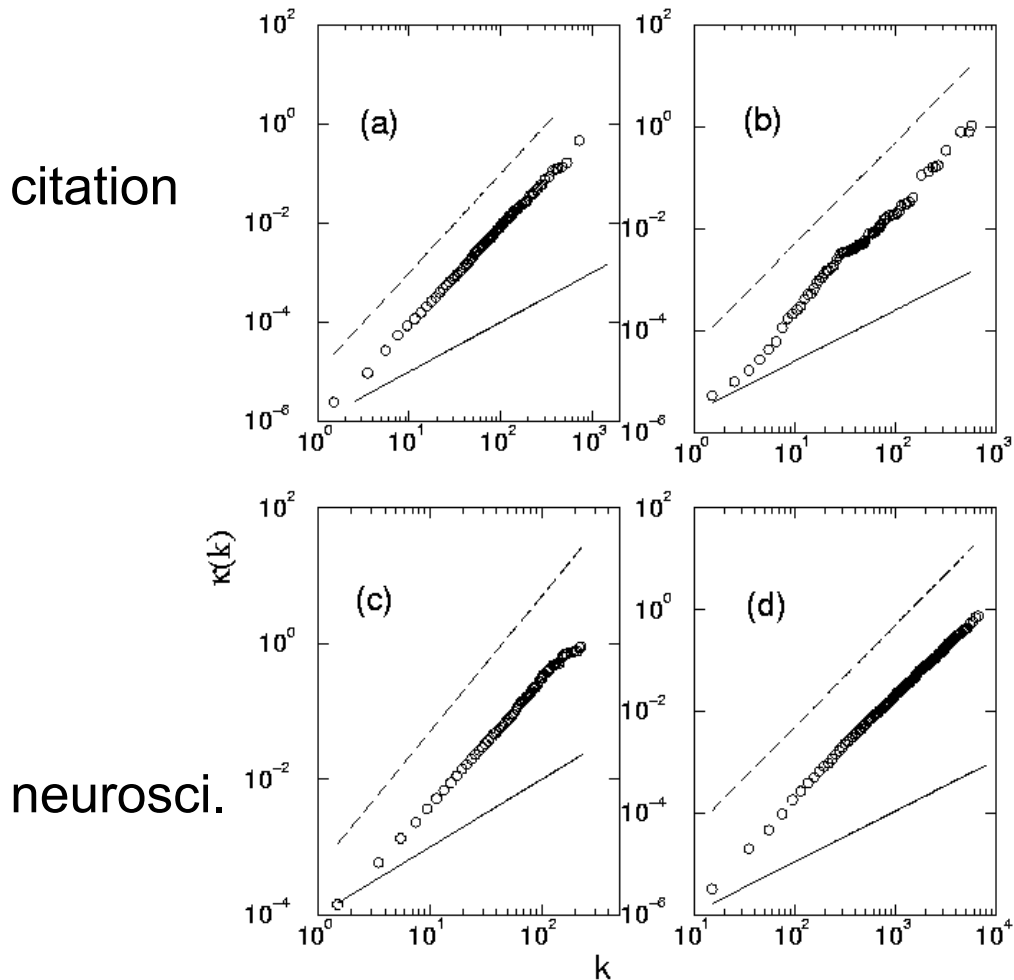
- addition of edges without new nodes

- edge rewiring, removal

- node removal

- constraints or optimization principles

Preferential attachment in real networks



$$\kappa(k) = \sum_{K < k} \Pi(K)$$

Internet

—— no pref. attach

----- linear pref. attach

$$\Pi(k) \approx A + k^\alpha, \quad \alpha \leq 1$$

actor collab.

Consequences of nonlinear preferential attachment

$$\Pi(k) \approx A + k^\alpha, \quad \alpha \leq 1$$

A - initial attractiveness

1. Sublinear preferential attachment leads to a stretch-exponential degree distribution.

$$P(k) \approx \exp\left(-\left(k/k_0\right)^\beta\right)$$

P. Krapivsky, S. Redner, F. Leyvraz, Phys. Rev. Lett. 85, 4629 (2000)

2. Initial attractiveness only shifts the degree exponent.

$$\gamma_{in} = 2 + \frac{A}{m} \quad \text{directed network, starting point is 2}$$

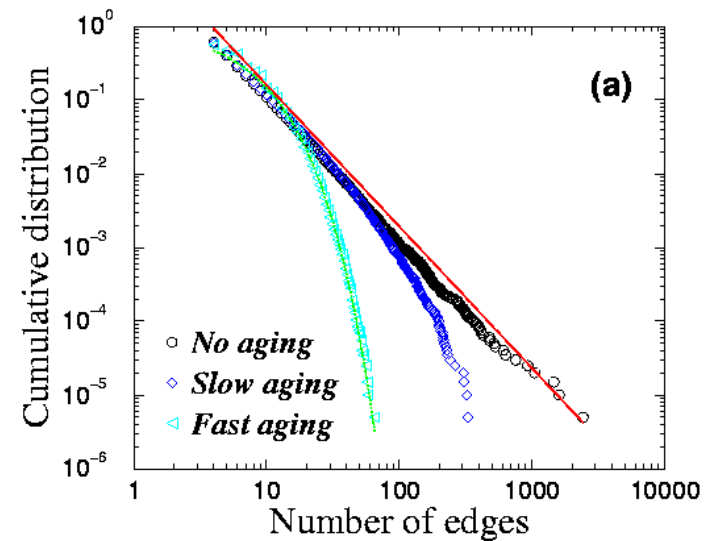
Dorogovtsev, Mendes, Samukhin, Phys. Rev. Lett. 85, 4633 (2000)

Mechanisms for preferential attachment

1. Copying mechanism
 - directed network
 - select a node and an edge of this node
 - attach to the endpoint of this edge
2. Walking on a network
 - directed network
 - the new node connects to a node, then to every first, second, ... neighbor of this node
3. Attaching to edges
 - select an edge
 - attach to both endpoints of this edge
4. Node duplication
 - duplicate a node with all its edges
 - randomly prune edges of new node

Growth constraints and aging cause cutoffs

- Finite lifetime to acquire new edges



L. A. N. Amaral et al., PNAS 97, 11149 (2000)

- Gradual aging: $\Pi(k_i) \propto k_i (t - t_i)^{-\nu}$
 γ increases with ν

S. N. Dorogovtsev and J. F. F. Mendes, Phys. Rev. E 62, 1842 (2000)

Additional processes also change the degree exponent

- mp new edges
 - mq edges rewired
- $$P(k) \approx (k + k_0)^{-\gamma} \quad k_0, \gamma = f(p, q, m)$$

R. Albert, A.-L. Barabási, Phys. Rev. Lett 85, 5234 (2000)

- c edges added or removed

$$\gamma_{in} = 2 + \frac{1}{1 + 2c}$$

S. N. Dorogovtsev, J. F. F. Mendes, Europhys. Lett. 52, 33 (2000)

Example of an evolving network model

At each step a choice is made between three possible mechanisms

- m new edges, one node selected randomly, the other with preferential attachment
done with probability p

$$\Pi(k_i) = \frac{k_i + 1}{\sum_j (k_j + 1)}$$

- m edges rewired – disconnected from a randomly selected node and connected to a node using pref. attach.
done with probability q
- a new node added with m edges that connect to nodes selected with preferential attachment
done with probability $1-p-q$

Rate equation approach

$$\begin{aligned}
 \frac{dN_k}{dt} = & \underbrace{pm \frac{1}{N} N_{k-1}}_{k-1 \rightarrow k, \text{ rand}} + m \underbrace{\frac{k}{\sum_j k_j} N_{k-1}}_{k-1 \rightarrow k, \text{ pref}} \\
 & - \underbrace{(p+q)m \frac{1}{N} N_k}_{k \rightarrow k+1, \text{ rand}} - m \underbrace{\frac{k+1}{\sum_j (k_j+1)} N_k}_{k \rightarrow k+1, \text{ pref}} \\
 & + \underbrace{qm \frac{1}{N} N_{k+1}}_{k+1 \rightarrow k, \text{ rand}} + \delta_{k,m}
 \end{aligned}$$

Continuum theory

$$\frac{dk_i}{dt} = \underbrace{(p-q)m \frac{1}{N}}_{\text{random edge gain}} + \underbrace{m \frac{k_i + 1}{\sum_j (k_j + 1)}}_{\text{random edge loss}} + \underbrace{m \frac{k_i + 1}{\sum_j (k_j + 1)}}_{\text{preferential edge gain}}$$

$$P(k) \approx (k + k_0(p, q, m))^{-\gamma(p, q, m)}$$

$$k_0(p, q, m) = (p - q) \left(\frac{2m(1 - q)}{1 - p - q} + 1 \right) + 1$$

$$\gamma(p, q, m) = \frac{2m(1 - q) + 1 - p - q}{m} + 1$$

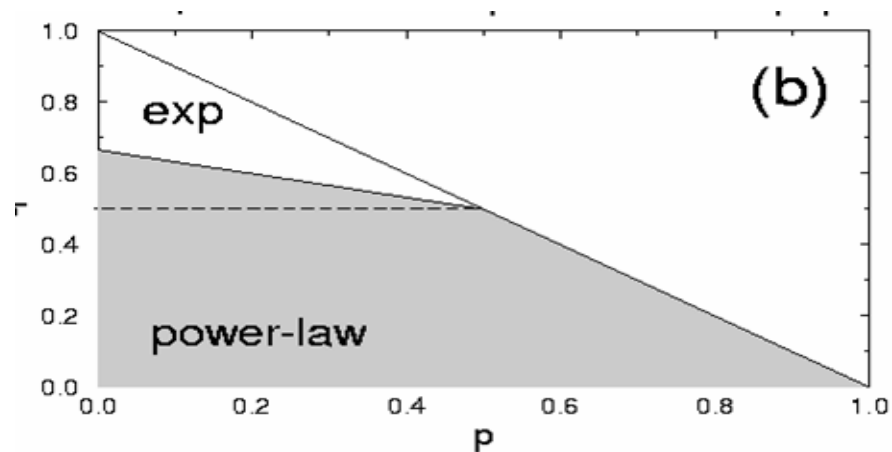
R. Albert, A.-L. Barabási, Phys. Rev. Lett 85, 5234 (2000)

Continuum theory

$$P(k) \approx (k + k_0(p, q, m))^{-\gamma(p, q, m)}$$

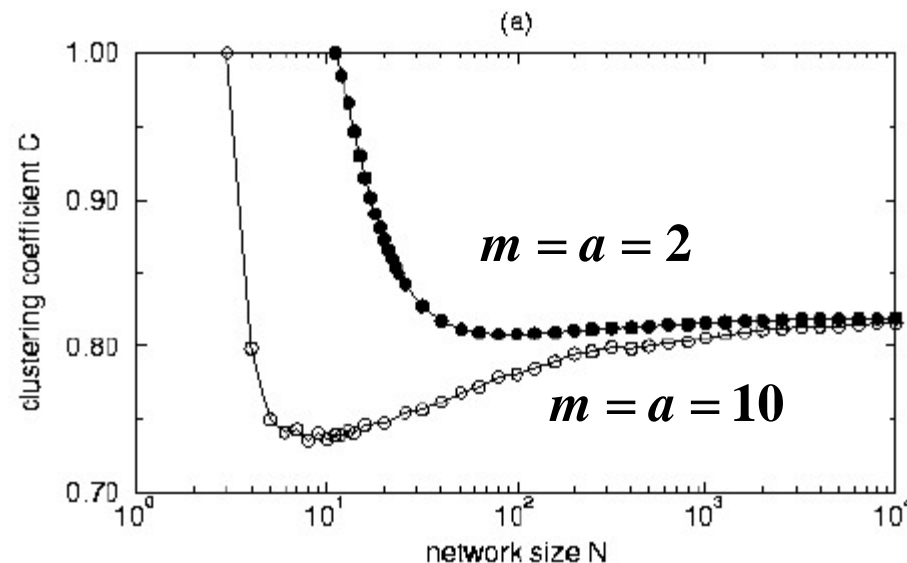
only valid if $(p - q) \left(\frac{2m(1 - q)}{1 - p - q} + 1 \right) + m + 1 > 0$

Elsewhere the degree distribution becomes an exponential



A model with high clustering coefficient

- Each node of the network can be either **active** or **inactive**.
- There are only m active nodes in the network at any instance.
 1. Start with m active, completely connected nodes
 2. At each timestep add a new node (active) that connects to m active nodes.
 3. Deactivate one active node $P_d(k_i) \propto (a + k_j)^{-1}$



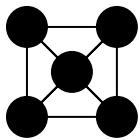
$$\Pi(k) \approx a + k$$

$$P(k) \approx k^{-2-a/m}$$

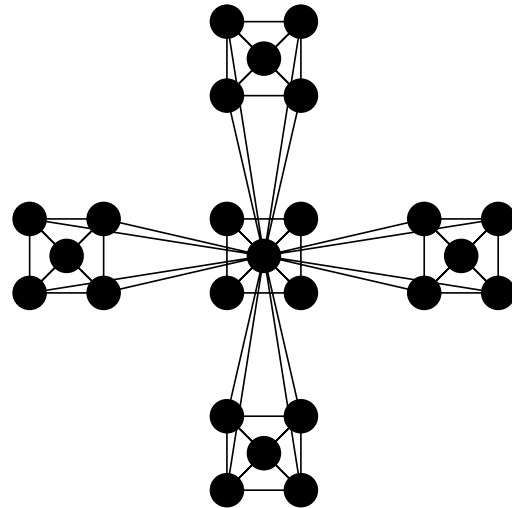
K. Klemm and V. Eguiluz, Phys. Rev. E 65, 036123 (2002)

A deterministic scale-free model

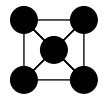
- Start with a completely connected graph with five nodes (one “central” , four peripheral
- Make four copies of the graph, keep the original in the center. Connect the four peripheral nodes of each copy to the central node of the original.
- Make four copies of the graph, again connect peripheral nodes to the central node.



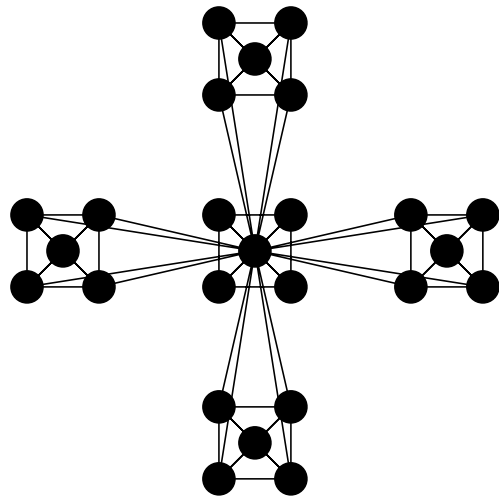
5-clique



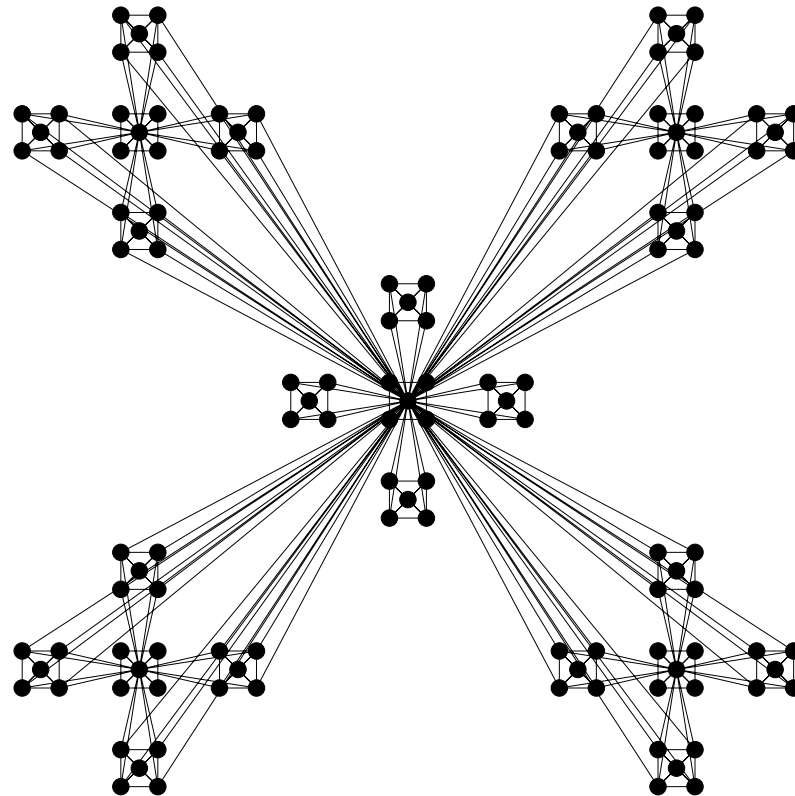
A deterministic scale-free model



5-clique



connect peripheries
to central node



E. Ravasz, A.-L. Barabasi, Phys Rev E 67, 026112 (2003)

Ex. 1

How does the number of nodes increase as a function of time steps?

Ex. 2

How does the degree of the central node increase in time?

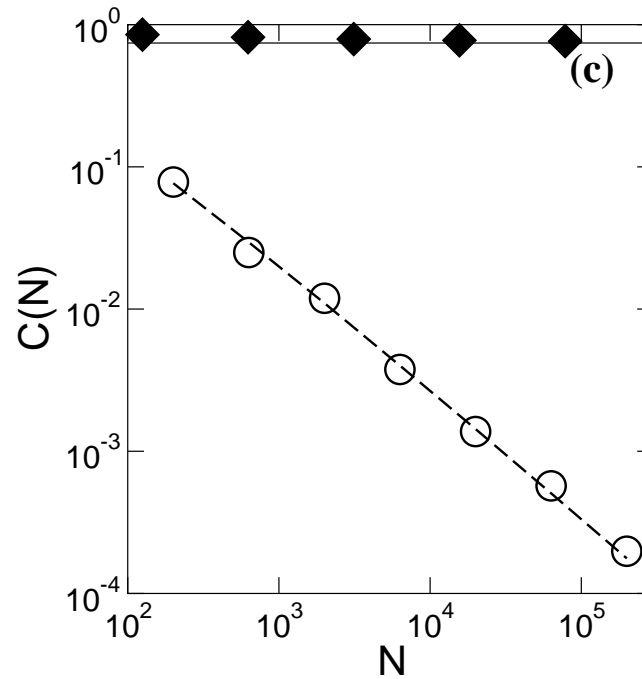
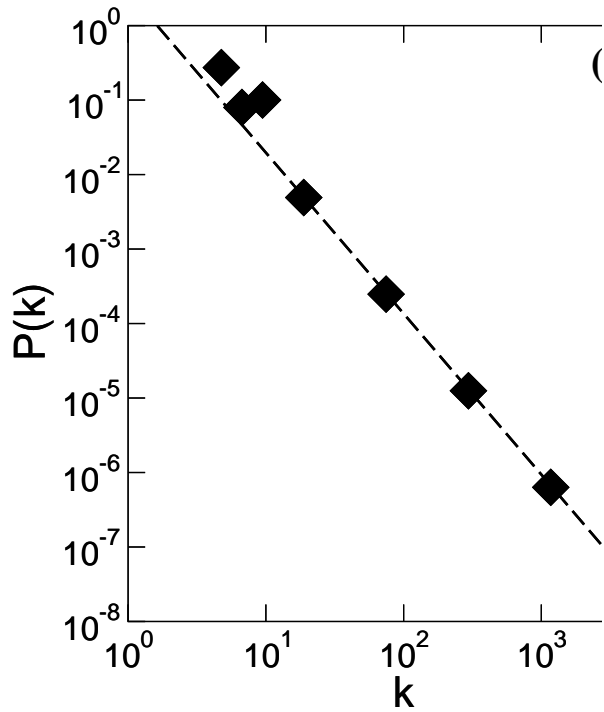
Ex. 3

How does the number of edges increase as a function of time steps?

Ex. 4

Can you identify the highest degree nodes?

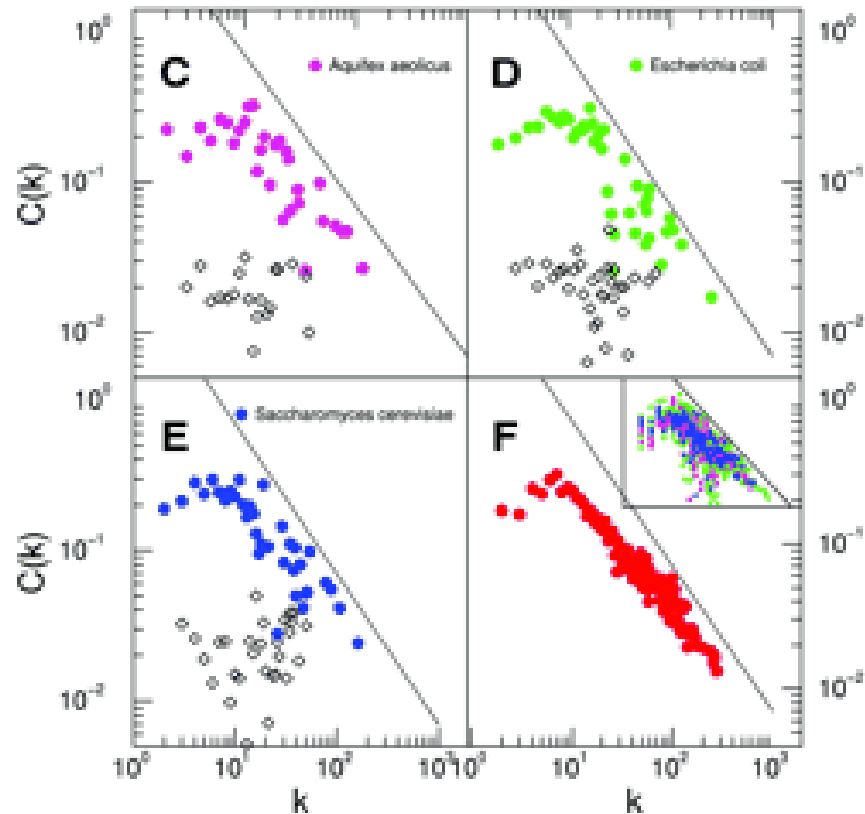
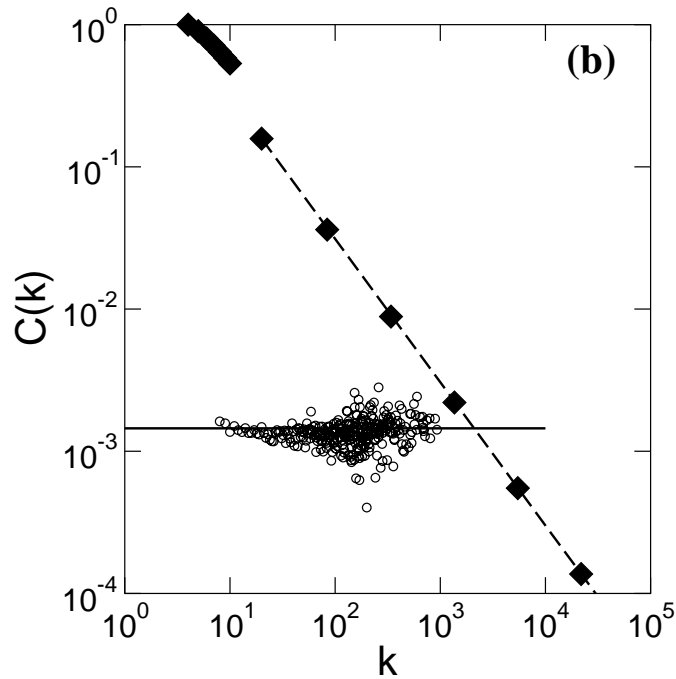
Properties of the model



Degree distribution $P(k) \propto k^{-1-\ln 4 / \ln 3}$

Clustering coefficient $C \approx 0.6$ independent of network size

Hierarchical structure



Average clustering coefficient of nodes with degree k

$$C(k) \approx k^{-1}$$

Also observed in various cellular networks – sign of hierarchical, modular architecture

E. Ravasz et al., Science 297, 1551 (2002)

| | | |
|--|--|--|
| Linear growth, linear pref. attachment | $\gamma=3$ | Barabási and Albert, 1999 |
| Nonlinear preferential attachment $\Pi(k_i) \sim k_i^\alpha$ | no scaling for $\alpha \neq 1$ | Krapivsky, Redner, and Leyvraz, 2000 |
| Asymptotically linear pref. attachment $\Pi(k_i) \sim a_\infty k_i$ as $k_i \rightarrow \infty$ | $\gamma \rightarrow 2$ if $a_\infty \rightarrow \infty$ $\gamma \rightarrow \infty$ if $a_\infty \rightarrow 0$ | Krapivsky, Redner, and Leyvraz, 2000 |
| Initial attractiveness $\Pi(k_i) \sim A + k_i$ | $\gamma=2$ if $A=0$ $\gamma \rightarrow \infty$ if $A \rightarrow \infty$ | Dorogovtsev, Mendes, and Samukhin, 2000a, 2000b |
| Accelerating growth $\langle k \rangle \sim t^\theta$ constant initial attractiveness | $\gamma=1.5$ if $\theta \rightarrow 1$ $\gamma \rightarrow 2$ if $\theta \rightarrow 0$ | Dorogovtsev and Mendes, 2001a |
| Internal edges with probab. p | $\gamma=2$ if $q = \frac{1-p+m}{1+2m}$ | |
| Rewiring of edges with probab. q | $\gamma \rightarrow \infty$ if $p, q, m \rightarrow 0$ | Albert and Barabási, 2000 |
| c internal edges or removal of c edges | $\gamma \rightarrow 2$ if $c \rightarrow \infty$ $\gamma \rightarrow \infty$ if $c \rightarrow -1$ | Dorogovtsev and Mendes, 2000c |
| Gradual aging $\Pi(k_i) \sim k_i(t-t_i)^{-\nu}$ | $\gamma \rightarrow 2$ if $\nu \rightarrow -\infty$ $\gamma \rightarrow \infty$ if $\nu \rightarrow 1$ | Dorogovtsev and Mendes, 2000b |
| Multiplicative node fitness $\Pi_i \sim \eta_i k_i$ | $P(k) \sim \frac{k^{-1-c}}{\ln(k)}$ | Bianconi and Barabási, 2001a Dorogovtsev, Mendes, and Samukhin, 2000c |
| Edge inheritance $P(k_{in}) = \frac{d}{k_{in}^{\sqrt{2}}} \ln(ak_{in})$ | | |
| Copying with probab. p | $\gamma = (2-p)/(1-p)$ | Kumar <i>et al.</i> , 2000a, 2000b |
| Redirection with probab. r | $\gamma = 1 + 1/r$ | Krapivsky and Redner, 2001 |
| Walking with probab. p | $\gamma \approx 2$ for $p > p_c$ | Vázquez, 2000 |
| Attaching to edges | $\gamma=3$ | Dorogovtsev, Mendes, and Samukhin, 2001a |
| p directed internal edges $\Pi(k_i, k_j) \propto (k_i^{in} + \lambda)(k_j^{out} + \mu)$ | $\gamma_{in} = 2 + p\lambda$ $\gamma_{out} = 1 + (1-p)^{-1} + \mu p / (1-p)$ | Krapivsky, Rodgers, and Redner, 2001 |

Lessons learned from evolving network models

1. There is no universal exponent characterizing all networks.
2. The origins of the preferential attachment might be system-dependent.
3. It is generally true that networks *evolve*.
4. Modeling real networks:
 - identify the processes that play a role
 - measure their frequency from real data
 - develop dynamical models that capture these processes